INTUITIONISTIC MODAL ALGEBRAS

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ABSTRACT. Recent research on algebraic models of *quasi-Nelson logic* has brought new attention to a number of classes of algebras which result from enriching (subreducts of) Heyting algebras with a special modal operator, known in the literature as a *nucleus*. Among these various algebraic structures, for which we employ the umbrella term *intuitionistic modal algebras*, some have been studied since at least the 1970s, usually within the framework of topology and sheaf theory. Others may seem more exotic, for their primitive operations arise from algebraic terms of the intuitionistic modal language which have not been previously considered. We shall for instance investigate the variety of *weak implicative semilattices*, whose members are (non-necessarily distributive) meet semilattices endowed with a nucleus and an implication operation which is not a relative pseudo-complement but satisfies the postulates of Celani and Jansana's strict implication. For each of these new classes of algebras we establish a representation and a topological duality which generalize the known ones for Heyting algebras enriched with a nucleus.

Keywords: intuitionistic modal algebras; nuclei; representation; topological duality; nuclear Heyting algebras; implicative semilattices; quasi-Nelson algebras; fragments; weak Heyting algebras.

1. INTRODUCTION

A *nuclear Heyting algebra* is obtained by enriching a Heyting algebra $\langle H; \wedge, \vee, \to, 0, 1\rangle$ with a unary modal operator $\Box$ satisfying the following identity:

$$x \to \Box y = \Box x \to \Box y.$$

(One can equivalently require $\Box$ to satisfy either the properties stated in Definition 2.5 or those of Definition 2.6; see below.) Such an operator is also known in the literature as a *nucleus* or as a *multiplicative closure operator*[1]. Many natural constructions give rise to nuclei. For instance, having fixed an element $a \in H$ of a Heyting algebra, we can obtain a nucleus by setting either $\Box x := a \to x$ or $\Box x := a \vee x$, or $\Box x := (x \to a) \to a$. So, in particular, the identity map, the constant map $x \mapsto 1$ and the double negation map also define nuclei (see [18, 1] for further examples).

The class of nuclear Heyting algebras (and some of its subreducts) has been studied since the 1970s, usually within the framework of topology and sheaf theory [18, 20, 3, 2]. A more recent paper [14] proposed a logic based on nuclear Heyting algebras (called *Lax Logic*) as a tool in the formal verification of computer hardware. Even more recently, another connection between nuclear Heyting algebras and logic emerged within the study of the algebraic semantics of *quasi-Nelson logic* [30, 29]. The latter may be viewed as a common generalization of both intuitionistic logic and *Nelson's constructive logic with strong negation* [22] obtained by deleting the double negation law.

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[1]The notion of *nucleus* (on a Heyting algebra) can be traced back at least to the early 1970s. Under the name of 'modal operators', nuclei are studied extensively in [18], which refers to earlier work (from the late 1960s) by F.W. Lawvere.

As shown in [29, 26, 25], there exists a formal relation between the algebraic counterpart of quasi-Nelson logic and the class of nuclear Heyting algebras which parallels the well-known connection between *Nelson algebras* and Heyting algebras (see e.g. [32]). This relation – which, as we shall see, concerns the algebras in the full language as well as some of their subreducts – provides, in our view, further motivation for the study of nuclear Heyting algebras from a logical as well as an algebraic point of view. It is interesting to note that, with the notable exception of [1], studies of this kind are scant in the literature – perhaps owing to the mainly topological interest in this class of algebras. The purpose of the present contribution is to fill in this gap, at least partly, and at the same time to draw attention to certain subreducts of nuclear Heyting algebras whose interest is motivated by the recent developments in the theory of quasi-Nelson logic.

Since a nuclear Heyting algebra is usually presented in the language $\{\wedge, \vee, \rightarrow, \Box, 0, 1\}$, fragments that appear to be of natural interest (from a logico-algebraic perspective) are, for instance, the implication-free one $\{\wedge, \vee, \Box\}$ – perhaps enriched with the lattice bounds 0 and 1 – and the implicational one $\{\rightarrow, \Box\}$. The former, whose models are distributive lattices enriched with a modal operator, is in fact the main object of [1], while the latter – whose models are *Hilbert algebras*, the algebraic counterpart of the purely implicational fragment of intuitionistic logic, expanded with a modal operator – was studied, mainly from a topological perspective, as far back as in [18], and as recently as in [12]. Other less obvious but, in our opinion, also interesting classes of algebras emerged in the course of our recent investigations on quasi-Nelson logic and its algebraic counterpart, the variety of *quasi-Nelson algebras*. An interest in these classes of algebras, however, can also be motivated within the limits of the traditional framework of nuclear Heyting algebras, as we shall try to explain below.

A well-known fact in the theory of nuclear Heyting algebras [18, Thm. 2.12] is that, for every such algebra $\mathbf{H} = \langle H; \wedge, \vee, \rightarrow, \Box, 0, 1\rangle$, the set $H_\Box := \{a \in H : a = \Box a\}$ of fixpoints of the $\Box$ operator can itself be endowed with a nuclear Heyting algebra structure by defining, for every n-ary algebraic operation $f \in \{\wedge, \vee, \rightarrow, \Box, 0, 1\}$, the operation $f_\Box$ given, for all $a_1, \ldots, a_n \in H_\Box$, by:

$$f_\Box(a_1, \ldots, a_n) := \Box f(a_1, \ldots, a_n).$$

Denoting this algebra by $\mathbf{H}_\Box$, we observe that the universe $H_\Box$ can equivalently be defined as the nucleus image $\{\Box a : a \in H\}$ of $\mathbf{H}$. While $\mathbf{H}_\Box$ is indeed a nuclear Heyting algebra, it is a very special one on which the $\Box$ operator is the identity map. This very fact, in turn, is essential in ensuring that $\mathbf{H}_\Box$ has a Heyting algebra reduct; for instance we have, for all $a, b \in H_\Box$,

$$a \wedge_\Box b = \Box(a \wedge b) = \Box a \wedge \Box b = a \wedge b$$

guaranteeing that $\wedge_\Box$ is a meet semilattice operation on $H_\Box$. A similar reasoning applies to the other operations, although the join $\vee_\Box$ (computed in $\mathbf{H}_\Box$) does not coincide with the join $\vee$ (computed in $\mathbf{H}$), i.e. $\mathbf{H}_\Box$ is not a subalgebra of $\mathbf{H}$. This construction is easily seen to be a generalization of Glivenko's result relating Heyting and Boolean algebras (the latter corresponding to the case where $\Box x = \neg\neg x$).

Thus, although nothing prevents one from considering each operation $f_\Box$ as defined on the whole universe H, in general $\wedge_\Box$ and $\vee_\Box$ will not be semilattice operations on H, and $\rightarrow_\Box$ will not be a Heyting (i.e. relative pseudo-complement) implication on H (on the other hand, we always have $\Box_\Box = \Box$ and $1_\Box = 1$. By definition, these new operations will be generalizations of the intuitionistic ones, which can be retrieved by requiring $\Box$ to be the identity map on H. In this respect natural questions to ask are, in our opinion, (1) which properties each generalized operation $f_\Box$ retains, and (2) whether some particular choice of $f_\Box$ has any independent interest that may justify further study.

A first answer to the latter question may be sought within the theory of quasi-Nelson logic. Indeed, as shown in the papers [29, 26, 25, 27], some of the above-defined operations of type $f_\Box$ naturally arise within the study of fragments of the quasi-Nelson language. From this standpoint, it is also interesting to observe that the classes of algebras one obtains through the *twist representation* (see below) combine the original Heyting operations with the new ones. Thus, for instance, one of the classes of algebras arising in this way (see Definition 3.1) retains the original meet semilattice operation (and the lattice bounds) while replacing the Heyting implication with a generalized counterpart: that is, we are looking at the $\{\wedge, \to_\Box, 0, 1\}$-subreducts of nuclear Heyting algebras. We stress that these new algebras are not the result of an arbitrary choice of operations, but arise as twist factors in the representation of subreducts of quasi-Nelson algebras, as we now proceed to explain.

A *quasi-Nelson algebra* may be defined as a commutative integral bounded residuated lattice (see e.g. [16] for formal definitions of these terms) $\mathbf{A} = \langle A; \sqcap, \sqcup, *, \Rightarrow, \bot\rangle$ that (upon letting $\sim x := x \Rightarrow \bot$) satisfies the *Nelson identity*:

$$(x \Rightarrow (x \Rightarrow y)) \sqcap (\sim y \Rightarrow (\sim y \Rightarrow \sim x)) = x \Rightarrow y.$$

Quasi-Nelson algebras arise as the algebraic counterpart of quasi-Nelson logic, which can be viewed either as a generalization (i.e. a weakening) common to Nelson's constructive logic with strong negation and to intuitionistic logic, or as the extension (i.e. strengthening) of the well-known substructural logic FL_{ew} (the *Full Lambek Calculus with Exchange and Weakening*) by the *Nelson axiom*:

$$((x \Rightarrow (x \Rightarrow y)) \sqcap (\sim y \Rightarrow (\sim y \Rightarrow \sim x))) \Rightarrow (x \Rightarrow y).$$

We refer to [30] for further details on quasi-Nelson logic, as well as for other equivalent characterizations of the variety of quasi-Nelson algebras (which can e.g. also be obtained as the class of $(0,1)$-*congruence orderable* commutative integral bounded residuated lattices).

Formally, every Heyting algebra may be viewed as a quasi-Nelson algebra (on which $\wedge = *$, $\vee = \sqcup$, $\to = \Rightarrow$ and $0 = \bot$) and, as noted earlier, the double negation map defines a modal operator on every Heyting algebra $\mathbf{H}$. If we replace $\mathbf{H}$ by a quasi-Nelson algebra $\mathbf{A}$, then the double negation map still gives us a nucleus in the sense of Definition 2.5 but no longer in the sense of Definition 2.6, for item (ii) may fail to be satisfied (indeed, the equivalence of both definitions breaks down outside the setting of Heyting algebras). The double negation can, however, be used to obtain a nucleus on a special quotient $H(\mathbf{A})$, which is the (Heyting) algebra canonically associated to each quasi-Nelson algebra $\mathbf{A}$ via the twist construction.

Given a quasi-Nelson algebra $\mathbf{A}$, consider the map given, for all $a \in A$, by $a \mapsto a * a$. The kernel θ of this map is a congruence of the reduct $\langle A; \sqcap, \sqcup, *\rangle$ which is also compatible with the double negation operation and with the *weak implication* $\Rightarrow^2$ given by $x \Rightarrow^2 y := x \Rightarrow (x \Rightarrow y)$. Letting $\Box(x/\theta) := \sim\sim x/\theta$, we thus have a quotient algebra $H(\mathbf{A}) = \langle A/\theta; \sqcap, \sqcup, \Rightarrow^2, \Box, \bot\rangle$, which is a nuclear Heyting algebra (where $* = \sqcap$). Moreover, $\mathbf{A}$ embeds into a *twist-algebra* over $H(\mathbf{A})$, defined as follows.

Definition 1.1. Let $\mathbf{H} = \langle H; \wedge, \vee, \to, \Box, 0, 1\rangle$ be a nuclear Heyting algebra. Define the algebra $\mathbf{H}^{\bowtie} = \langle H^{\bowtie}; \sqcap, \sqcup, *, \Rightarrow, \bot\rangle$ with universe:

$$H^{\bowtie} := \{\langle a_1, a_2\rangle \in H \times H_\Box : a_1 \wedge a_2 = 0\}$$

and operations given, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in H \times H$, by:

$$\begin{aligned}
\bot &:= \langle 0, 1\rangle \\
\langle a_1, a_2\rangle * \langle b_1, b_2\rangle &= \langle a_1 \wedge b_1, \Box((a_1 \to b_2) \wedge (b_1 \to a_2))\rangle \\
\langle a_1, a_2\rangle \sqcap \langle b_1, b_2\rangle &:= \langle a_1 \wedge b_1, \Box(a_2 \vee b_2)\rangle \\
\langle a_1, a_2\rangle \sqcup \langle b_1, b_2\rangle &:= \langle a_1 \vee b_1, \Box(a_2 \wedge b_2)\rangle \\
\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle &:= \langle (a_1 \to b_1) \wedge (b_2 \to a_2), \Box(a_1 \wedge b_2)\rangle.
\end{aligned}$$

A *quasi-Nelson twist-algebra over* $\mathbf{H}$ is any subalgebra $\mathbf{A} \leq \mathbf{H}^{\bowtie}$ satisfying $\pi_1[A] = H$.

The *twist representation theorem* says that every quasi-Nelson algebra $\mathbf{A}$ embeds into the twist-algebra $(H(\mathbf{A}))^{\bowtie}$ through the map given by $a \mapsto \langle a/\theta, \sim a/\theta\rangle$ for all $a \in A$ [30].

Definition 1.1 suggests that certain term operations of the language of nuclear Heyting algebras may be of particular interest in the study of fragments of the quasi-Nelson language. Consider, for instance, the monoid operation $(*)$. In order to define it, on a quasi-Nelson algebra $\mathbf{A} \leq \mathbf{H}^{\bowtie}$, we need two operations on $\mathbf{H}$: the semilattice operation $\wedge$ (for the first component) and, for the second component, an implication-like operation (henceforth denoted by $\rightharpoonup$) which can be given by $x \rightharpoonup y := x \to \Box y$. The latter claim may not be obvious, but using the properties of the twist construction and the modal operation, it is not hard to verify the following equalities:

$$\begin{aligned}
\Box((a_1 \to b_2) \wedge (b_1 \to a_2)) &= \Box((a_1 \to \Box b_2) \wedge (b_1 \to \Box a_2)) \\
&= \Box(a_1 \to \Box b_2) \wedge \Box(b_1 \to \Box a_2) \\
&= (a_1 \to \Box b_2) \wedge (b_1 \to \Box a_2) \\
&= (a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2).
\end{aligned}$$

These observations led to the introduction of the class of algebras dubbed $\rightharpoonup$*-semilattices* in [27], where it is shown in particular that the $\{*, \sim\}$-subreducts of quasi-Nelson algebras are precisely the algebras representable as twist-algebras over $\rightharpoonup$-semilattices. Similar considerations motivated the introduction of other term operations of the language of nuclear Heyting algebras, such as the following:

$$x \odot y := \Box(x \wedge y) \qquad x \oplus y := \Box(x \vee y).$$

As shown in [27], the corresponding classes of modal algebras (see Definitions 3.9 and 3.14) allow us to establish twist representations for (respectively) the classes of $\{\Rightarrow^2, \sim\}$-subreducts and of $\{\wedge, *, \Rightarrow, \sim\}$-subreducts of quasi-Nelson algebras. Other subreducts may be obtained by adding a modal operator to more traditional classes of intuitionistic algebras, such as implicative semilattices (corresponding to the $\{*, \Rightarrow, \sim\}$-subreducts of quasi-Nelson algebras), distributive lattices (corresponding to the $\{\wedge, \vee, \sim\}$-subreducts studied in [28]) and pseudo-complemented lattices (corresponding to the “two-negations” subreducts studied in [26]).

The previous considerations suggest the above-mentioned classes of modal algebras as mathematical objects that may be of interest both in themselves and in relation to the study of non-classical logics, in particular Nelson's logics[2]. The aim of the present paper is to improve our understanding of these classes of algebras from an algebraic and a topological point of view.

The rest of the paper is organized as follows. Section 2 recalls the definitions of (subreducts of) Heyting algebras and of the modal operators known as *nuclei*.

[2]Beyond the Nelson realm, (0-free subreducts of prelinear) nuclear Heyting algebras also feature in the twist-type representation introduced in [17] for *Sugihara monoids*, a variety of algebras related to relevance logics.

In Section 3 we introduce the main classes of algebras of interest. The first is the variety of *weak implicative semilattices* (Subsection 3.1), which is as a variety of semilattices enriched with an implication operation $\rightharpoonup$ that, while not being necessarily a relative pseudo-complement, satisfies the postulates of Visser's strict implication [9]. We then introduce the variety of *nuclear Hilbert semigroups* (Subsection 3.2), whose members consist of bounded Hilbert algebras (the purely implicational subreducts of Heyting algebras) enriched with a pseudo-meet operation $\odot$ giving rise to a semigroup that is not necessarily a semilattice. The variety of *nuclear implicative semilattices* (Subsection 3.3), which is not new (see e.g. [2]), mostly interests us as a basis for our definition of $\oplus$*-implicative semilattices* (Subsection 3.4): the latter are bounded implicative semilattices (i.e. the join-free subreducts of Heyting algebras) endowed with a nucleus and with a join-like operation $\oplus$ which forms a semigroup but not necessarily a semilattice.

In Section 4 we recall or establish some simple facts about congruences and homomorphisms of the above-mentioned classes of algebras; these will be useful for characterizing the morphisms in the corresponding categories.

In Section 5 we introduce topological dualities for our new classes of intuitionistic modal algebras: in order to do so we shall build on the existing dualities for *Lax Hilbert algebras* (i.e. Hilbert algebras expanded with a nucleus, Subsection 5.1) and for (non-necessarily distributive) meet semilattices (Subsection 5.3). The new dualities are established in Subsections 5.2, 5.4 and 5.5.

Finally, the concluding Section 6 contains some indications for future research.

2. Heyting algebras, subreducts and nuclei

In this section we recall the main definitions of subreducts of Heyting algebras that are relevant to our study, as well as the definition(s) of nucleus.

The purely implicational subreducts of Heyting algebras are known in the literature as *Hilbert algebras*, or *(positive) implication algebras*.

Definition 2.1. A *Hilbert algebra* is an algebra $\langle H; \to, 1\rangle$ of type $\langle 2, 0\rangle$ that satisfies the following (quasi-)identities:

(i) $x \to (y \to x) = 1$.
(ii) $x \to (y \to z) = (x \to y) \to (x \to z)$.
(iii) if $x \to y = y \to x = 1$, then $x = y$.

Every Hilbert algebra has a natural order $\leq$ given, for all $a, b \in H$, by $a \leq b$ iff $a \to b = 1$, having 1 as top element (as a matter of fact, the constant 1 need not be included in the language, for it is term definable by $1 := x \to x$). If the order $\leq$ also has a minimum element (denoted 0), we speak of a *bounded Hilbert algebra*. In such a case we include 0 in the algebraic signature, and one can define a *negation* operation $\neg$ by $\neg x := x \to 0$. Bounded Hilbert algebras correspond to the $\{\to, \neg, 0, 1\}$-subreducts of Heyting algebras.

The subreducts of Heyting algebras obtained by retaining only the infimum and the negation operations form the class of *pseudo-complemented semilattices*, or *p-semilattices* [15, 31].

Definition 2.2. A *pseudo-complemented semilattice* (*p-semilattice*) is an algebra $\langle S; \wedge, \neg, 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$ such that:

(i) $\langle S; \wedge, 0, 1\rangle$ is a bounded semilattice (with order $\leq$).

(ii) $x \leq \neg y$ (i.e. $x \wedge \neg y = x$) if and only if $x \wedge y = 0$.

A *pseudo-complemented lattice* (*p-lattice*) is an algebra $\langle L; \wedge, \vee, \neg, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ such that $\langle L; \wedge, \neg, 0, 1\rangle$ is a *p*-semilattice and $\langle L; \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice.

We shall refer to item (ii) above as to the "property of the pseudo-complement". Pseudo-complemented semilattices form a variety whose only proper subvariety is the class of Boolean algebras [31, p. 305]; the latter can thus be relatively axiomatized by adding any identity that is not valid on all pseudo-complemented semilattices (for instance the involutive law $\neg\neg x = x$).

If we retain both the meet and the intuitionistic implication, we obtain *implicative semilattices* (also known as *Brouwerian semilattices*).

Definition 2.3. An *implicative semilattice* is an algebra $\langle S; \wedge, \to, 1\rangle$ of type $\langle 2, 2, 0\rangle$ such that:

(i) $\langle S; \wedge, 1\rangle$ is an upper-bounded semilattice (with order $\leq$ and top element 1).
(ii) $x \wedge y \leq z$ if and only if $x \leq y \to z$.

The property in item (ii) is known as *residuation*, and we shall say that $\langle \wedge, \to\rangle$ is a *residuated pair*. Implicative meet semilattices are precisely the $\vee$-free subreducts of Heyting algebras; in turn, the $\wedge$-free reduct of every implicative meet semilattice forms a Hilbert algebra. A *bounded implicative semilattice* is one whose semilattice reduct has a least element 0. In such a case, by letting $\neg x := x \to 0$, one obtains a pseudo-complemented semilattice. With the above definitions in mind, Heyting algebras may be introduced as follows.

Definition 2.4. A *Heyting algebra* is an algebra $\langle H; \wedge, \vee, \to, 0, 1\rangle$ of type $\langle 2, 2, 2, 0, 0\rangle$ such that:

(i) $\langle H; \wedge, \vee, 0, 1\rangle$ is a bounded lattice.
(ii) $\langle H; \wedge, \to, 1\rangle$ is an implicative semilattice.

The pseudo-complement negation $\neg$ is defined, on every Heyting algebra, by $\neg x := x \to 0$, as in the preceding cases.

In the following sections we shall consider algebras that result from adding a modal-like operator to the subreducts of Heyting algebras introduced earlier. Such operators are known as *nuclei* (or *modal operators*, or *multiplicative closure operators*), and have been extensively studied in the literature on residuated lattices and Heyting algebras; for our purposes, the results contained in the dissertation [18] will be particularly useful. We shall consider two different but essentially equivalent definitions for a nucleus, which depend on which other operations are available on the algebra.

Definition 2.5. Let **A** be an algebra having a reduct $\langle A; \wedge, 0\rangle$ that is a (meet) semilattice with order $\leq$ and minimum 0. We shall say that an operation $\Box \colon A \to A$ is a *nucleus* on **A** if the following identities are satisfied:

(i) $x \leq \Box x = \Box\Box x$
(ii) $\Box(x \wedge y) = \Box x \wedge \Box y$

If $\Box 0 = 0$, then we say that $\Box$ is *dense*.

Observe that the above properties entail that, if the order $\leq$ has a maximum element 1, then $\Box 1 = 1$ (so the $\Box$ operator is indeed modal-like in that it preserves all finite meets). When the underlying algebra does not have a meet operation, we can define a nucleus as follows.

Definition 2.6 ([25], Def. 4.3)**.** Given an algebra having a bounded Hilbert algebra reduct $\langle H; \to, 0, 1\rangle$, we say that an operation $\Box\colon H \to H$ is a *nucleus* on $\mathbf{H}$ if:

(i) $x \leq \Box x = \Box\Box x$,
(ii) $\Box(x \to y) \leq \Box x \to \Box y$,

As before, if $\Box 0 = 0$, then we say that $\Box$ is *dense*.

Following [12], a Hilbert algebra endowed with a nucleus, thus viewed as an algebra in the language $\{\to, \Box, 1\}$, will be called a *Lax Hilbert algebra* (an *LH-algebra*, for short).

3. Intuitionistic modal algebras

In this section we introduce a number of classes of algebras that, as observed earlier, arise from the twist representation of (subreducts of) quasi-Nelson algebras. All these carry a nucleus operator together with one or more algebraic operations that are term definable in the language of Heyting algebras enriched with the nucleus.

3.1. **Weak implicative semilattices.** As mentioned in the introduction, the algebras introduced in the next definition were first considered in [27], under the name of $\rightharpoonup$-*semilattices*, as factors in the twist representation of $\{*, \sim\}$-subreducts of quasi-Nelson algebras. Here we introduce the more suggestive term "weak implicative semilattices", which we shall explain in a moment.

Definition 3.1. A *weak implicative semilattice* (abbreviated WIS) is an algebra $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$ satisfying the following properties (we abbreviate $\Box x := 1 \rightharpoonup x$):

(i) $\langle S; \wedge, 0, 1\rangle$ is a bounded semilattice (with order $\leq$).
(ii) $x \rightharpoonup (y \rightharpoonup z) = (x \wedge y) \rightharpoonup z$.
(iii) $x \rightharpoonup (y \wedge z) = (x \rightharpoonup y) \wedge (x \rightharpoonup z)$.
(iv) $\Box 0 = 0$.
(v) $x \leq \Box x$.
(vi) $x \wedge \Box y = x \wedge (x \rightharpoonup y)$.
(vii) $x \leq y \rightharpoonup z$ if and only if $x \wedge y \leq \Box z$.
(viii) $x \rightharpoonup y = \Box x \rightharpoonup \Box y$.

Items (ii)–(v) of the preceding Definition entail that the operation $\Box$ given by $\Box x := 1 \rightharpoonup x$ indeed realizes a nucleus (in the sense of Definition 2.5) on every weak implicative semilattice $\mathbf{S}$. Therefore, whenever convenient, we shall consider weak implicative semilattices as algebras in the language that includes the nucleus thus defined.

The operation $\rightharpoonup$ can be thought of as a generalized (intuitionistic) implication in the following sense. For every algebra having a bounded implicative semilattice reduct $\langle S; \wedge, \to, 0, 1\rangle$ as per Definition 2.3 and a nucleus $\Box$, we can obtain a weak implicative semilattice by letting $x \rightharpoonup y := x \to \Box y$ (cf. Definition 3.12 and Example 3.13). As a nucleus we can for example take the double negation (which gives us $x \rightharpoonup y = x \to \neg\neg y$) or the identity function on S (which gives us $\rightharpoonup\, = \,\to$).

The following remark should explain the name chosen for our algebras. Consider the class of *weak Heyting algebras* (or WH-algebras), which arose in [9] from the study of strict implication fragments of modal logics. Formally, a WH-algebra is a bounded distributive lattice $\langle L; \wedge, \vee, 0, 1\rangle$ further endowed with a binary operation $\rightharpoonup$ which satisfies the following identities [9, Def. 3.1]:

(WH1) $x \rightharpoonup (y \wedge z) = (x \rightharpoonup y) \wedge (x \rightharpoonup z)$.

(WH2) $(x \vee y) \rightharpoonup z = (x \rightharpoonup z) \wedge (y \rightharpoonup z)$.

(WH3) $(x \rightharpoonup y) \wedge (y \rightharpoonup z) \leq x \rightharpoonup z$.

(WH4) $x \rightharpoonup x = 1$.

It is not hard to verify that every weak implicative semilattice satisfies all the above identities, except of course (WH2). Conversely, the $\vee$-free reduct of every WH-algebra satisfies all the properties listed in Definition 3.1 except perhaps item (v). Those WH-algebras that satisfy item (v) are known as *basic algebras* [9, Def. 3.3], and form the algebraic counterpart of Visser's logic [34].

On every weak implicative semilattice $\mathbf{S}$ (or, more generally, on every algebra having a nucleus $\Box$), we can consider the set of $\Box$-fixpoints, which can be given in either of the following ways:

$$S_\Box := \{a \in S : a = \Box a\} = \{\Box a : a \in S\}.$$

It is easy to verify that, for every weak implicative semilattice $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$, the set $S_\Box$ is the universe of a subalgebra

$$\mathbf{S}_\Box = \langle S_\Box; \wedge, \rightharpoonup, 0, 1\rangle,$$

which is a bounded implicative semilattice. The case where $S_\Box = S$ is characterized in the following proposition.

Proposition 3.2 ([27], Prop. 3.2)**.** *Let $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$ be a weak implicative semilattice. The following are equivalent:*

(i) $\mathbf{S} \models \Box x \leq x$.

(ii) $\langle S; \rightharpoonup, 0, 1\rangle$ *is a (bounded) Hilbert algebra.*

(iii) $\langle S; \wedge, \rightharpoonup, 0, 1\rangle$ *is a (bounded) implicative semilattice.*

Regarding the preceding proposition, we observe that the algebras on which the nucleus is the identity function (which are then just ordinary subreducts of Heyting algebras) are of special interest within the theory of quasi-Nelson algebras, for they correspond precisely to subreducts of Nelson algebras (i.e. involutive quasi-Nelson algebras).

The next propositions better explain the relationship between weak implicative semilattices and p-semilattices.

Proposition 3.3 ([27], Prop. 3.3)**.** *Given a pseudo-complemented semilattice $\mathbf{P} = \langle P; \wedge, \neg, 0, 1\rangle$, define $x \rightharpoonup y := \neg(x \wedge \neg y)$. Then $\langle P; \wedge, \rightharpoonup, 0, 1\rangle$ is a weak implicative semilattice.*

As observed in [31, Cor. 4.14], the congruence lattice of a pseudo-complemented semilattice need not satisfy any non-trivial lattice identity. Proposition 3.3 thus entails that the same holds for weak implicative semilattices; in particular, they are in general non-distributive semilattices (cf. Section 4).

Proposition 3.4 ([27], Cor. 3.5)**.** *Let $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$ be a weak implicative semilattice. Upon defining $\neg x := x \rightharpoonup 0$, the algebra $\langle S; \wedge, \neg, 0, 1\rangle$ is a pseudo-complemented semilattice.*

Proposition 3.5 ([27], Prop. 3.6)**.** *Let $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$ be a weak implicative semilattice, with the pseudo-complement operation $\neg$ given by $\neg x := x \rightharpoonup 0$. The following are equivalent:*

(i) $\mathbf{S} \models \Box x = \neg\neg x$.

(ii) $\mathbf{S} \models x \rightharpoonup y = \neg(x \wedge \neg y)$.

It may be interesting to notice that $\neg(x \wedge \neg y)$ is precisely the term that defines the "classical" implication within D. Prawitz's *Ecumenical System*, a calculus designed for combining classical and intuitionistic logic (see e.g. [24]); likewise the term interpreting the "classical" disjunction within Prawitz's system matches the one given for the operation $\oplus$ that we shall consider in Example 3.15.

As expected, every implicative semilattice with a nucleus $\mathbf{S} = \langle S; \wedge, \to, \Box, 0, 1\rangle$ (Definition 3.12) may be endowed with a weak implicative semilattice structure by letting $x \rightharpoonup y := x \to \Box y$. Given this definition, we have the following result, which was used in [27] to show that the class of twist-algebras over weak implicative semilattice structure coincides with the $\{*, \sim\}$-subreducts of quasi-Nelson algebras.

Proposition 3.6 ([27], Prop. 6.2)**.** *Every weak implicative semilattice embeds into a complete*[3] *implicative semilattice with a nucleus.*

It was shown in [2] that the class of bounded nuclear implicative semilattices is locally finite. This result, together with Proposition 3.6, entails that *weak implicative semilattices are also locally finite.*

Proposition 3.7. *The class of weak implicative semilattices is a variety.*

Proof. It suffices to verify that item (vii) of Definition 3.1, which is the one non-equational condition, can be equivalently replaced by the following two:

(I) $x \rightharpoonup x = 1$.
(II) $x \leq y \rightharpoonup x$.

It is easy to verify that (I) and (II) are satisfied by all weak implicative semilattices. Indeed, by item (vii) of Definition 3.1, we have $1 \leq x \rightharpoonup x$ iff $1 \wedge x \leq \Box x$, which is true by item (v). Similarly, we have $x \leq y \rightharpoonup x$ iff $x \wedge y \leq x$, which is certainly true.

For the other direction of the equivalence, let $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$ be an algebra that satisfies all items of Definition 3.1 except perhaps (vii). Given elements $a, b, c \in S$, assume $a \leq b \rightharpoonup c$. Then $a \wedge b \leq b \wedge (b \rightharpoonup c)$, and item (vi) of Definition 3.1 gives us $b \wedge (b \rightharpoonup c) = b \wedge \Box c \leq \Box c$, from which the result easily follows.

Conversely, assume $a \wedge b \leq \Box c$. The latter means that $a \wedge b = a \wedge b \wedge \Box c$. Then $b \rightharpoonup (a \wedge b) = b \rightharpoonup (a \wedge b \wedge \Box c)$, and by item (iii) of Definition 3.1 and (I), we have $b \rightharpoonup (a \wedge b) = (b \rightharpoonup a) \wedge (b \rightharpoonup b) = (b \rightharpoonup a) \wedge 1 = b \rightharpoonup a$. Likewise, $b \rightharpoonup (a \wedge b \wedge \Box c) = (b \rightharpoonup a) \wedge (b \rightharpoonup b) \wedge (b \rightharpoonup \Box c) = (b \rightharpoonup a) \wedge (b \rightharpoonup \Box c)$. Hence $b \rightharpoonup a = (b \rightharpoonup a) \wedge (b \rightharpoonup \Box c)$, which by (II) gives us $a \leq b \rightharpoonup a \leq b \rightharpoonup \Box c = b \rightharpoonup c$, as required. To justify the last equality, recall that $\Box$ is a nucleus, and in particular $\Box\Box c = \Box c$, because (by item (ii) of Definition 3.1) $\Box\Box c = 1 \rightharpoonup (1 \rightharpoonup c) = (1 \wedge 1) \rightharpoonup c = 1 \rightharpoonup c = \Box c$. Then, using item (viii) of Definition 3.1, we have $b \rightharpoonup \Box c = \Box b \rightharpoonup \Box\Box c = \Box b \rightharpoonup \Box c = b \rightharpoonup c$. ☐

Remark 3.8. As observed earlier, for each weak implicative semilattice $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$, the nucleus image $S_\Box$ is the universe of a subalgebra $\mathbf{S}_\Box = \langle S_\Box; \wedge, \rightharpoonup, 0, 1\rangle$ which is a bounded implicative semilattice. Conversely, from a pair $\langle \mathbf{S}, \mathbf{I}\rangle$, where $\mathbf{S}$ is a bounded meet semilattice and $\mathbf{I}$ is a bounded implicative semilattice related by suitable maps (if one wishes, I may also simply be taken to be a subset of S), we can obtain a weak implicative semilattice as follows. Let $\mathbf{S} = \langle S; \wedge_{\mathbf{S}}, 0_{\mathbf{S}}, 1_{\mathbf{S}}\rangle$ and $\mathbf{I} = \langle I; \wedge_{\mathbf{I}}, \to_{\mathbf{I}}, 0_{\mathbf{I}}, 1_{\mathbf{I}}\rangle$ be as above and let $n \colon S \to I$ and $p \colon I \to S$ be maps satisfying the following properties:

[3] As per standard terminology, a semilattice $\langle S; \wedge, 0, 1\rangle$ is *complete* when the meet of every subset $B \subseteq A$ (denoted $\bigwedge B$) exists in A.

(i) both n and p preserve finite meets and the bounds;
(ii) $s \leq_{\mathbf{S}} p(n(s))$ for all $s \in S$;
(iii) $Id_I = n \circ p$.

Then, letting $x \rightharpoonup y := p(n(x) \to n(y))$, we have that $\langle S; \wedge_{\mathbf{S}}, \rightharpoonup, 0_{\mathbf{S}}, 1_{\mathbf{S}} \rangle$ is a weak implicative semilattice. Indeed, one can prove that every weak implicative semilattice $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1 \rangle$, arises in this way from the pair $\langle \langle S; \wedge, 0, 1 \rangle, \langle S_\Box; \wedge, \rightharpoonup, 0, 1 \rangle \rangle$ by letting $n = \Box$ and $p = Id_{S_\Box}$.

3.2. **nH-semigroups.** The class of algebras considered in this section was introduced to provide a twist representation for the (weak) implication-negation subreducts of quasi-Nelson algebras [25, 29]; the latter class, in turn, is the equivalent algebraic semantics of the algebraizable fragment of quasi-Nelson logic studied in [19].

We recall that, according to the twist construction of quasi-Nelson algebras (Definition 1.1), the weak implication is given by:

$$\langle a_1, a_2 \rangle \Rightarrow^2 \langle b_1, b_2 \rangle = \langle a_1 \to b_1, \Box(a_1 \wedge b_2) \rangle.$$

This suggests that the factor algebras corresponding to the weak implication-negation subreducts will need to carry, besides the nucleus operator, only an intuitionistic implication $\to$ and a "pseudo-meet" operation $\odot$ such that $x \odot y = \Box(x \wedge y)$; this motivates the following definition.

Definition 3.9 ([25], Def. 4.5)**.** A *bounded nuclear Hilbert semigroup* (nH-semigroup for short) is an algebra $\mathbf{S} = \langle S; \odot, \to, 0, 1 \rangle$ such that:

(i) $\langle S; \to, 0, 1 \rangle$ is a bounded Hilbert algebra.
(ii) $\langle S; \odot \rangle$ is a commutative semigroup.
(iii) The operation $\Box$ given by $\Box x := x \odot x$ is a dense nucleus on $\langle S; \to, 0, 1 \rangle$ in the sense of Definition 2.6.
(iv) $x \odot y = x \odot (x \to y)$.
(v) $\Box x \to (\Box y \to z) = (x \odot y) \to z$.
(vi) $x \odot 0 = 0$.
(vii) $x \odot 1 = \Box x$.

It is clear that the $\{\to, \Box, 1\}$-reduct of every nH-semigroup is a Lax Hilbert algebra in the sense of [12], an observation that we shall exploit later on. Observe that nH-semigroups generalize bounded implicative semilattices, for every bounded implicative semilattice $\langle A; \wedge, \to, 0, 1 \rangle$ is an nH-semigroup where $\wedge = \odot$ and $\Box$ is the identity map. As in the case of weak implicative semilattices, it is also easy to verify that the algebra of $\Box$-fixpoints $\langle S_\Box; \odot, \to, 0, 1 \rangle$ of every nH-semigroup $\mathbf{S} = \langle S; \odot, \to, 0, 1 \rangle$ is a bounded implicative semilattice. The following examples provide some further insight on nH-semigroups.

Example 3.10 (cf. [25], Prop. 4.8)**.** Let $\mathbf{A}$ be any algebra having a bounded Hilbert algebra reduct $\langle A; \to, 0, 1 \rangle$. Define $\neg x := x \to 0$ and $x \odot y := \neg(x \to \neg y)$. Then the algebra $\langle A; \odot, \to 0, 1 \rangle$ is an nH-semigroup where $\Box x = \neg\neg x$.

Lemma 3.11 (cf. [25], Lemma 4.9)**.** *Let* $\mathbf{S} = \langle S; \odot, \to, 0, 1 \rangle$ *be an nH-semigroup and* $a, b, c \in S$.

(i) $\Box(a \odot b) = \Box a \odot \Box b = a \odot b$.
(ii) If $a \leq b$*, then* $\Box a = a \odot b$.
(iii) $a \to (b \odot c) = (a \to \Box b) \odot (a \to \Box c)$.

(iv) $a \to 0 = \Box a \to 0$.

(v) $a \odot b \leq \Box a$.

Proof. Items (i), (ii) and (iv) match those of [25, Lemma 4.9]. Item (iii) is slightly more general than it appears on [25, Lemma 4.9], namely: $\Box a \to (b \odot c) = (\Box a \to \Box b) \odot (\Box a \to \Box c)$. But the two formulations are easily seen to be equivalent by the identities $x \to \Box y = \Box x \to \Box y$ [25, Lemma 4.4] and $x \to (y \odot z) = x \to \Box(y \odot z)$, which is a consequence of item (i). Finally, regarding (v), using Definition 3.9 (v) we have $(a \odot b) \to \Box a = \Box a \to (\Box b \to \Box a) = 1$. □

3.3. **Nuclear implicative semilattices.** The following definition is easily seen to be equivalent to the one adopted in [2] which is based, for the nucleus, on our Definition 2.5.

Definition 3.12. A *bounded nuclear implicative semilattice* is an algebra $\mathbf{S} = \langle S; \wedge, \to, \Box, 0, 1 \rangle$ such that:

(i) $\langle S; \wedge, \to, 0, 1 \rangle$ is a bounded implicative semilattice;
(ii) $\Box$ is a nucleus (Definition 2.6) on the bounded Hilbert algebra reduct $\langle S; \to, 0, 1 \rangle$.

As in the preceding cases, the algebra of $\Box$-fixpoints $\mathbf{S}_\Box = \langle S_\Box; \wedge, \to, 0, 1 \rangle$ of a nuclear implicative semilattice $\mathbf{S} = \langle S; \wedge, \to, \Box, 0, 1 \rangle$ is a bounded implicative semilattice. Thus each class of algebras K introduced so far is *nuclear* in the sense of [3], that is, for every member $\mathbf{A} \in \mathsf{K}$, we have $\mathbf{A}_\Box \in \mathsf{K}$. The following example should help clarify the relationship among the above-mentioned classes.

Example 3.13 (cf. [25], Lemma 4.6)**.** Let $\mathbf{S}$ be any algebra having a reduct $\langle S; \wedge, \to, \Box, 0, 1 \rangle$ that is a bounded nuclear implicative semilattice. Define $x \rightharpoonup y := x \to \Box y$ and $x \odot y := \Box x \wedge \Box y$. Then the algebra $\langle S; \wedge, \rightharpoonup, 0, 1 \rangle$ is a weak implicative semilattice (Definition 3.1) and the algebra $\langle S; \odot, \to 0, 1 \rangle$ is an nH-semigroup (Definition 3.9). In particular, by taking $\Box$ to be the identity map, we have that $\langle S; \wedge, \to, 0, 1 \rangle$ is both a weak implicative semilattice and an nH-semigroup.

3.4. **$\oplus$-implicative semilattices.** As mentioned earlier, nH-semigroups arise as factors in the twist representation of the weak implication-negation subreducts of quasi-Nelson algebras. If we enrich the latter with the quasi-Nelson meet operation, we obtain a variety of algebras (dubbed *quasi-Nelson semihoops* in [27]) which can be represented as twist-algebras over the class of $\oplus$-implicative semilattices introduced below. Indeed, Definition 1.1 suggests that, in order to represent the quasi-Nelson meet ($\sqcap$), one only needs to introduce a further binary operation (here denoted $\oplus$) such that $x \oplus y = \Box(x \vee y)$.

As observed in [27], the quasi-Nelson monoid operation ($*$) is definable as follows:

$$x * y := x \wedge y \wedge {\sim}((x \Rightarrow^2 {\sim} y) \wedge (y \Rightarrow^2 {\sim} x)).$$

Thus every quasi-Nelson semihoop also carries the quasi-Nelson monoid operation; this is in keeping with the observation that, according to Definition 1.1, the operation $*$ can be defined on twist-algebras using only the implicative semilattice operations (and the nucleus) of the factor algebra.

Definition 3.14. A *$\oplus$-implicative semilattice* is an algebra $\mathbf{S} = \langle S; \wedge, \oplus, \to, 0, 1 \rangle$ such that:

(i) $\langle S; \wedge, \to, \Box, 0, 1 \rangle$ is a bounded nuclear implicative semilattice whose nucleus is given by $\Box x := x \oplus x$ (Definition 3.12).
(ii) $\langle S; \oplus \rangle$ is a commutative semigroup.

(iii) $x \oplus 1 = 1$.
(iv) $\Box x = x \oplus 0 = x \oplus (x \wedge y)$.
(v) $x \leq x \oplus y = \Box x \oplus \Box y$.
(vi) $\Box x \wedge (y \oplus z) = (x \wedge y) \oplus (x \wedge z)$.

Let us illustrate the preceding definition with an example.

Example 3.15. Let $\langle H; \wedge, \vee, \to, \Box, 0, 1\rangle$ be a nuclear Heyting algebra. Then, upon defining $x \oplus y := \Box(x \vee y)$, the algebra $\langle H; \wedge, \oplus, \to, 0, 1\rangle$ is a $\oplus$-implicative semilattice. Thus, in particular, every Heyting algebra may be viewed as a $\oplus$-implicative semilattice where, taking the nucleus $\Box$ to be the identity map, we have that $\oplus$ coincides with the lattice join, whereas taking $\Box$ to be the double negation map we have $x \oplus y = \neg(\neg x \wedge \neg y)$.

The class of $\oplus$-implicative semilattices is also nuclear in the above-introduced sense: given $\mathbf{S} = \langle S; \wedge, \oplus, \to, 0, 1\rangle$, we have that the algebra of $\Box$-fixpoints $\mathbf{S}_{\Box} = \langle S_{\Box}; \wedge, \oplus, \to, 0, 1\rangle$ is a Heyting algebra (so $\mathbf{S}_{\Box}$ is also a $\oplus$-implicative semilattice).

Given a nuclear Heyting algebra $\langle H; \wedge, \vee, \to, \Box, 0, 1\rangle$, letting $x \oplus y := \Box(x \vee y)$, we can recall the following result from [27], which was used to characterize the class of $\{\sqcap, \Rightarrow^2, \sim\}$-subreducts of quasi-Nelson algebras.

Proposition 3.16 ([27], Lemma 6.6)**.** *Every $\oplus$-implicative semilattice embeds into a complete nuclear Heyting algebra.*

The following properties will be useful later on, in particular for giving a characterization of congruences of $\oplus$-implicative semilattices.

Lemma 3.17 ([27], Lemma 5.5)**.** *Every $\oplus$-implicative semilattice $\mathbf{S} = \langle S; \wedge, \oplus, \to, 0, 1\rangle$ satisfies the following (quasi-)identities.*

(i) If $x \leq z$ and $y \leq z$, then $x \oplus y \leq \Box z$.
(ii) $\Box(x \oplus y) = x \oplus y$.
(iii) $x \to y \leq (x \oplus z) \to (y \oplus z)$.
(iv) $\neg(x \oplus y) = \neg x \wedge \neg y$.
(v) $(x \to \Box z) \wedge (y \to \Box z) \leq (x \oplus y) \to \Box z$.

Proof. The only item not proved in [27, Lemma 5.5] is the last one. Let $a, b, c \in S$. Since $(a \to \Box c) \wedge (b \to \Box c) \leq \Box(a \to \Box c) \wedge \Box(b \to \Box c)$, it suffices to show that $\Box(a \to \Box c) \wedge \Box(b \to \Box c) \leq (a \oplus b) \to \Box c$. To see this, let us compute:

$$\begin{array}{lll}
\Box(a \to \Box c) \wedge \Box(b \to \Box c) \leq (a \oplus b) \to \Box c & \text{iff} & \text{(by residuation)} \\
(a \oplus b) \wedge \Box(a \to \Box c) \wedge \Box(b \to \Box c) \leq \Box c & \text{iff} & \text{(by Def. 3.14.vi)} \\
((a \wedge (a \to \Box c)) \oplus (b \wedge (a \to \Box c))) \wedge \Box(b \to \Box c) \leq \Box c & \text{iff} & \text{(by } x \wedge (x \to y) = x \wedge y) \\
((a \wedge \Box c) \oplus (b \wedge (a \to \Box c))) \wedge \Box(b \to \Box c) \leq \Box c & \text{iff} & \text{(by Def. 3.14.vi)} \\
(a \wedge \Box c \wedge (b \to \Box c)) \oplus (b \wedge (a \to \Box c) \wedge (b \to \Box c)) \leq \Box c & \text{iff} & \text{(by } x \wedge (x \to y) = x \wedge y) \\
(a \wedge \Box c \wedge (b \to \Box c)) \oplus (b \wedge (a \to \Box c) \wedge \Box c) \leq \Box c & \text{iff} & \text{(by } x \leq y \to x) \\
(a \wedge \Box c) \oplus (b \wedge \Box c) \leq \Box c & \text{iff} & \text{(by Def. 3.14.vi)} \\
\Box\Box c \wedge (a \oplus b) \leq \Box c & \text{iff} & \text{(by } \Box x = \Box\Box x) \\
\Box c \wedge (a \oplus b) \leq \Box c. & &
\end{array}$$

□

4. Congruences and homomorphisms

In this section we take a look at congruences and homomorphisms of the above-introduced classes of intuitionistic modal algebras. As we shall see, similarly to the case of Heyting algebras endowed with a nucleus, in most cases the congruences of each intuitionistic modal algebra coincide with those of its purely implicational reduct. The next well-known fact (see e.g. [21]) will be a key ingredient in subsequent proofs.

Lemma 4.1 ([29], Lemma 36)**.** *Let $\mathbf{H} = \langle H; \to, 1\rangle$ be a Hilbert algebra, $a, b \in A$ and $\theta \in \mathrm{Con}(\mathbf{H})$. The following conditions are equivalent:*

(i) $\langle a, b\rangle \in \theta$.

(ii) $\langle a \to b, 1\rangle, \langle b \to a, 1\rangle \in \theta$.

Proposition 4.2. *Let $\mathbf{A}$ be either (i) a Lax Hilbert algebra, (ii) an nH-semigroup, (iii) a nuclear implicative semilattice or (iv) a $\oplus$-implicative semilattice. In all these cases, the congruences of $\mathbf{A}$ coincide with those of the Hilbert algebra reduct of $\mathbf{A}$.*

Proof. (i). Let $\mathbf{A} = \langle A; \to, \Box, 1\rangle$ be a Lax Hilbert algebra, and let θ be a congruence of the Hilbert algebra reduct $\langle A; \to, 1\rangle$. Assume $\langle a, b\rangle \in \theta$. Then $\langle a \to b, 1\rangle, \langle b \to a, 1\rangle \in \theta$, by Lemma 4.1. Since $a \to b \leq \Box a \to \Box b$, from $\langle a \to b, 1\rangle, \langle b \to a, 1\rangle \in \theta$ we obtain $\langle \Box a \to \Box b, 1\rangle, \langle \Box b \to \Box a, 1\rangle \in \theta$ (this holds in every Hilbert algebra: see [25, Prop. 4.14]). Then we obtain $\langle \Box a, \Box b\rangle \in \theta$ by applying again Lemma 4.1.

(ii). This was shown in [25, Prop. 4.14].

(iii). Let $\mathbf{A} = \langle A; \wedge, \to, \Box, 1\rangle$ be a nuclear implicative semilattice. Then the reduct $\langle A; \to, \Box, 1\rangle$ is a Lax Hilbert algebra, and by item (i) we know that $\mathrm{Con}(\langle A; \to, \Box\rangle) = \mathrm{Con}(\langle A; \to\rangle)$. To conclude the proof, it remains to show that $\mathrm{Con}(\langle A; \wedge, \to\rangle) = \mathrm{Con}(\langle A; \to\rangle)$. Let us then assume that $\theta \in \mathrm{Con}(\langle A; \to\rangle)$ and $\langle a, b\rangle \in \theta$. Thus, by Lemma 4.1, we have $\langle a \to b, 1\rangle, \langle b \to a, 1\rangle \in \theta$. Since the identity $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$ holds on every implicative semilattice [10], for all $c \in A$, we have $(a \wedge c) \to (b \wedge c) = ((a \wedge c) \to b) \wedge ((a \wedge c) \to c) = ((a \wedge c) \to b) \wedge 1 = (a \wedge c) \to b \geq a \to b$. As noted in item (i), this and the assumption $\langle a \to b, 1\rangle \in \theta$ give us $\langle (a \wedge c) \to (b \wedge c), 1\rangle \in \theta$. A similar reasoning shows that $\langle (b \wedge c) \to (a \wedge c), 1\rangle \in \theta$ as well, so we can apply Lemma 4.1 to conclude that $\langle a \wedge c, b \wedge c\rangle \in \theta$. Since we are in a semilattice, this is sufficient to establish that θ is compatible with $\wedge$, as required.

(iv). Now let $\mathbf{A} = \langle A; \wedge, \oplus, \to, 0, 1\rangle$ be a $\oplus$-implicative semilattice and $\theta \in \mathrm{Con}(\langle A; \to\rangle)$. We know by item (iii) above that θ is compatible with $\wedge$ (and with the nucleus $\Box$, which is given by $\Box x := x \oplus x$). To conclude the proof, it suffices to show that θ is compatible with $\oplus$ as well. To this end, assume $\langle a, b\rangle \in \theta$ and let $c \in A$. By Lemma 4.1, our assumptions imply $\langle a \to b, 1\rangle, \langle b \to a, 1\rangle \in \theta$. We claim that $\langle (a \oplus c) \to (b \oplus c), 1\rangle \in \theta$. To see this, observe that $a \to b \leq a \to (b \oplus c)$ and $c \to (b \oplus c) = 1$ hold by Definition 3.14 (v) Thus we have $\langle a \to (b \oplus c), 1\rangle, \langle c \to (b \oplus c), 1\rangle \in \theta$, which give us $\langle (a \to (b \oplus c)) \wedge (c \to (b \oplus c)), 1\rangle \in \theta$. By Lemma 3.17 (ii) we have $b \oplus c = \Box(b \oplus c)$, so we can apply Lemma 3.17 (v) to compute: $(a \to (b \oplus c)) \wedge (c \to (b \oplus c)) = (a \to \Box(b \oplus c)) \wedge (c \to \Box(b \oplus c)) \leq (a \oplus c) \to \Box(b \oplus c) = (a \oplus c) \to (b \oplus c)$. Thus, the assumption $\langle (a \to (b \oplus c)) \wedge (c \to (b \oplus c)), 1\rangle \in \theta$ gives us $\langle (a \oplus c) \to (b \oplus c), 1\rangle \in \theta$. A similar reasoning allows us to conclude $\langle (b \oplus c) \to (a \oplus c), 1\rangle \in \theta$, so (again by Lemma 4.1) we have $\langle a \oplus c, b \oplus c\rangle \in \theta$. Since the operation $\oplus$ is commutative, this is sufficient to establish that θ is compatible with $\oplus$, as required. ☐

Given an algebra $\mathbf{A}$ with a partial order $\leq$ and maximum 1, we shall say that an element $a \in A$ is the *penultimate element* of A if $a \neq 1$ and, for all $b \in A$ such that $b < 1$, it holds that $b \leq a$.

Lemma 4.3 ([4], Thm. 56, [25], Thm. 4.22)**.** *A Hilbert algebra* $\mathbf{A}$ *is subdirectly irreducible if and only if (the underlying order of)* $\mathbf{A}$ *has a penultimate element.*

From Proposition 4.2 and Lemma 4.3 we immediately obtain the following result.

Corollary 4.4. *Let* $\mathbf{A}$ *be either a Lax Hilbert algebra, an nH-semigroup, an implicative semilattice (with a nucleus) or a* $\oplus$*-implicative semilattice. Then* $\mathbf{A}$ *is subdirectly irreducible if and only if its underlying poset has a penultimate element.*

Congruences correspond, of course, to special (i.e. surjective) homomorphisms; thus the result of Proposition 4.2 does not necessarily extend to arbitrary homomorphisms. This, however, does hold for nH-semigroups, as shown below.

Proposition 4.5. *A map* $h\colon S \to S'$ *between nH-semigroups* $\mathbf{S}$ *and* $\mathbf{S}'$ *is a homomorphism (i.e. preserves* $\odot, \to, 0$ *and* 1*) if and only if* h *is a bounded Lax Hilbert algebra homomorphism between the corresponding bounded Lax Hilbert algebra reducts (i.e. preserves* $\Box, \to, 0$ *and* 1*).*

Proof. It obviously suffices to prove that every Lax Hilbert algebra homomorphism preserves the semigroup operation. Let $h\colon S \to S'$ be a Lax Hilbert algebra homomorphism, and let $a, b \in S$. We have:

$$
\begin{aligned}
(h(a) \odot h(b)) \to h(a \odot b) &= \Box h(a) \to (\Box h(b) \to h(a \odot b)) && \text{by Def. 3.9.v} \\
&= h(\Box a \to (\Box b \to (a \odot b))) && h \text{ preserves } \to, \Box \\
&= h((a \odot b) \to (a \odot b)) && \text{by Def. 3.9.v} \\
&= h(1) = 1.
\end{aligned}
$$

Thus, $h(a) \odot h(b) \leq h(a \odot b)$. On the other hand,

$$
\begin{aligned}
h(a \odot b) \to (h(a) \odot h(b)) &= (h(a \odot b) \to \Box h(a)) \odot (h(a \odot b) \to \Box h(b)) && \text{by Lemma 3.11.iii} \\
&= h((a \odot b) \to \Box a) \odot h((a \odot b) \to \Box b) && h \text{ preserves } \to, \Box \\
&= h(1) \odot h(1) && \text{by Lemma 3.11.v} \\
&= 1 \odot 1 = \Box 1 = 1 && \text{by Def. 3.9.vii.}
\end{aligned}
$$

So we get $h(a \odot b) \leq h(a) \odot h(b)$, as desired. $\Box$

From Proposition 4.5 and the observation that every implicative semilattice $\langle A; \wedge, \to, 0, 1\rangle$ is an nH-semigroup where $\wedge = \odot$ and $\Box$ is the identity map, it is easy to adapt the preceding result to (not necessarily bounded) implicative semilattices (with a nucleus).

Corollary 4.6. *A map* $h\colon S \to S'$ *between implicative semilattices* $\mathbf{S}$ *and* $\mathbf{S}'$ *is a homomorphism (i.e. preserves* $\wedge$ *and* $\to$*) if and only if* h *is a Hilbert algebra homomorphism between the corresponding Hilbert algebra reducts (i.e. preserves* $\to$*).*

Corollary 4.7. *A map* $h\colon S \to S'$ *between (bounded) nuclear implicative semilattices* $\mathbf{S}$ *and* $\mathbf{S}'$ *is a homomorphism (i.e. preserves* $\wedge, \to, \Box, 1$ *and, if present, the bottom element* 0*) if and only if* h *is a Lax Hilbert algebra homomorphism between the corresponding (bounded) Lax Hilbert algebra reducts (i.e. preserves* $\to, \Box, 1$ *and, if present, the bottom element* 0*).*

The preceding results suggest that, when we consider (most of) the preceding classes of algebra from a categorical point of view (see Section 5), the central notion we will have to look at with regards to morphisms will be that of Lax Hilbert algebras homomorphism. For weak implicative semilattices, we shall also consider the weaker notion of *semi-homomorphism*.

Definition 4.8. A map $h\colon S \to S'$ between weak implicative semilattices $\mathbf{S}$ and $\mathbf{S}'$ is a *semi-homomorphism* if, for all $a, b \in S$,

(SH1) $h(0) = 0$,

(SH2) $h(1) = 1$,

(SH3) $h(a \wedge b) = h(a) \wedge h(b)$,

(SH4) $h(a \rightharpoonup b) \leq h(a) \rightharpoonup h(b)$.

Thus a semi-homomorphism is a bounded meet semilattice homomorphism that further satisfies (SH4). We shall say that h is a *homomorphism* if h preserves all operations, that is, h is a semi-homomorphism that further satisfies:

(SH5) $h(a) \rightharpoonup h(b) \leq h(a \rightharpoonup b)$.

Lemma 4.9. *A map $h\colon S \to S'$ between weak implicative semilattices $\mathbf{S}$ and $\mathbf{S}'$ is a semi-homomorphism if and only if h is a bounded meet semilattice homomorphism such that $h(\Box a) \leq \Box h(a)$ for all $a \in S$.*

Proof. For the non-trivial direction, assume $h(\Box a) \leq \Box h(a)$, for all $a \in S$. Let $a, b \in S$. Since h is a meet semilattice homomorphism $h(a) \wedge h(a \rightharpoonup b) = h\left(a \wedge (a \rightharpoonup b)\right)$, and as h is monotonic, $h\left(a \wedge (a \rightharpoonup b)\right) \leq h(\Box b) \leq \Box h(b)$. Thus, $h(a) \wedge h(a \rightharpoonup b) \leq \Box h(b)$, i.e., $h(a \rightharpoonup b) \leq h(a) \rightharpoonup h(b)$. □

5. Dualities

We are now ready to obtain representations and topological dualities for the above-mentioned classes of intuitionistic modal algebras. We begin by recalling the main definitions and results from the duality for Lax Hilbert algebras introduced by Celani and Montangie in [11, 12] (Subsection 5.1); on this we shall build our duality for nH-semigroups (Subsection 5.2). Following a similar strategy, we shall use the duality for (implicative) meet semilattices developed in [7, 8, 13] by Celani and collaborators (Subsection 5.3) as a basis for our dualities for weak implicative (Subsection 5.4) and $\oplus$-implicative semilattices (Subsection 5.5).

Let $\langle X, \leq\rangle$ be a poset. For each $Y \subseteq X$, let $[Y) = \{x \in X : \exists y \in Y (y \leq x)\}$ and $(Y] = \{x \in X : \exists y \in Y (x \leq y)\}$. We will say that Y is an upset of X (a downset of X) if $Y = [Y)$ $(Y = (Y])$. We will write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively. We also write $\mathcal{P}(X)$ and $\mathrm{Up}(X)$ for the set of all subsets and upsets of$X$, respectively. We note that $\langle \mathrm{Up}(X), \cap, \cup, \emptyset, X\rangle$ is a bounded distributive lattice.

5.1. **Lax Hilbert algebras.**

The papers [11] and [12] introduce topological dualities for categories associated to Hilbert algebras with a modal operator, which can be easily extended to nH-semigroups.

Let us begin by introducing *Lax Hilbert spaces*, which are special T_0 spaces $\langle X, \tau_{\mathcal{K}}\rangle$ having a base of compact sets $\mathcal{K}$ enriched with a binary relation $R \subseteq X \times X$. Recall that the *saturation* of a set $Y \subseteq X$ is given by:

$$\mathrm{sat}(Y) := \bigcap \{U \in \mathcal{K} : Y \subseteq U\}$$

and the *closure* of $Y \subseteq X$ is given by:

$$\mathrm{cl}(Y) := \bigcap \{U^c : U \in \mathcal{K} \text{ and } Y \cap U = \emptyset\}.$$

We denote by $\leq_{\mathcal{K}}$, or by $\leq$, the *dual specialization order* given, for all $x, y \in X$, by $x \leq_{\mathcal{K}} y$ iff $y \in \mathrm{cl}(x)$. Recall that X is a T_0 space if and only if $\leq$ is a partial order. Recall also that a subset $Y \subseteq X$ is said to be *irreducible* when, for all closed sets $Y_1, Y_2 \subseteq X$, we have that $Y = Y_1 \cup Y_2$ entails $Y = Y_1$ or $Y = Y_2$. A space is *sober* when, for every irreducible closed set $Y \subseteq X$, there exists a unique $x \in X$ such that $Y = \mathrm{cl}(x)$.

Let X and Y be sets and $R \subseteq X \times Y$ a binary relation. For every $(x, y) \in X \times Y$, we consider the sets $R(x) = \{y \in X_2 : (x, y) \in R\}$, $R^{-1}(y) = \{x \in X_1 : (x, y) \in R\}$, and $R^{-1}(U) = \{x \in X : R(x) \cap U \neq \emptyset\}$, for $U \subseteq X$. We define the map $\Box_R : \mathcal{P}(Y) \to \mathcal{P}(X)$ given by

$$\Box_R(U) = \{x \in X : R(x) \subseteq U\}$$

for every $U \in \mathcal{P}(Y)$. Recall that $\Box_R(Y) = X$, and $\Box_R(U \cap V) = \Box_R(U) \cap \Box_R(V)$, for every $U, V \in \mathcal{P}(Y)$.

Definition 5.1 ([12], Def. 7)**.** A *Lax Hilbert space* (or *LH-space*) is a structure $\langle X, \tau_{\mathcal{K}}, Q\rangle$ such that:

(i) $\langle X, \tau_{\mathcal{K}}\rangle$ is a topological space having a base of compact sets $\mathcal{K}$.
(ii) $\mathrm{sat}(U \cap V^c) \in \mathcal{K}$ for all $U, V \in \mathcal{K}$.
(iii) $\langle X, \tau_{\mathcal{K}}\rangle$ is sober.
(iv) Q is a binary relation on X such that $Q(x)$ is a $\tau_{\mathcal{K}}$-closed set for all $x \in X$.
(v) $Q^{-1}(U) \in \mathcal{K}$ for all $U \in \mathcal{K}$.
(vi) $Q \subseteq \leq_{\mathcal{K}}$.
(vii) $Q = Q \circ Q$.

Definition 5.2 ([12], Def. 3; [11], Defs. 2.6 and 3.9)**.** Let $\langle X, \tau_{\mathcal{K}}, Q\rangle$ and $\langle X', \tau_{\mathcal{K}'}, Q'\rangle$ be LH-spaces. We say that $R \subseteq X \times X'$ is an *LH*-relation if:

(i) $R^{-1}(U) \in \mathcal{K}$ for all $U \in \mathcal{K}'$.
(ii) $R(x)$ is a $\tau_{\mathcal{K}'}$-closed set for all $x \in X$.

An *LH*-relation is *functional* whenever $\langle x, x'\rangle \in R$ entails $R(y) = [x')$ for some $y \in X$ with $x \leq y$.

Let LHSp denote the category of Lax Hilbert spaces with LH-relations, and let LHSpF be the subcategory of LHSp having Lax Hilbert spaces as objects and functional LH-relations as morphisms. Correspondingly, let LHA denote the category having Lax Hilbert algebras as objects and Lax Hilbert algebra *semi-homomorphisms* as morphisms (Definition 5.3).

Definition 5.3 ([12], Def. 3; [11], Defs. 2.6 and 3.9)**.** A map $h\colon H \to H'$ between Lax Hilbert algebras $\langle H; \to, \Box, 1\rangle$ and $\langle H'; \to', \Box', 1'\rangle$ is a *semi-homomorphism* if, for all $a, b \in H$,

(i) $h(1) = 1'$
(ii) $h(\Box a) = \Box' h(a)$
(iii) $h(a \to b) \leq' h(a) \to' h(b)$.

We denote by LHAH the subcategory of LHA having the same objects but requiring the morphisms to be algebraic homomorphisms, i.e. further satisfying $h(a \to b) = h(a) \to' h(b)$.

To obtain an equivalence, define functors $X\colon \mathsf{LHA} \to \mathsf{LHSp}$ and $H\colon \mathsf{LHSp} \to \mathsf{LHA}$ as follows.

Given a Lax Hilbert algebra $\langle H; \rightarrow, \Box \rangle$, we define an *(implicative) filter* as a non-empty subset $F \subseteq H$ that is closed under $\rightarrow$-*modus ponens*, that is, for all $a, b \in H$ such that $a \in F$ and $a \rightarrow b \in F$, we have $b \in F$. A filter F is *irreducible* when, for all filters F_1, F_2 such that $F = F_1 \cap F_2$, we have $F_1 = F$ or $F_2 = F$. The set of all irreducible filters of an algebra H is denoted by $X(H)$.

Consider the map σ given by $a \mapsto \{x \in X(H) : a \in X\}$ for all $a \in H$. Then the family:

$$\mathcal{K}_H := \{(\sigma(a))^c : a \in H\}$$

is the base for a topology $\tau_{\mathcal{K}_H}$. Further defining:

$$Q_H := \{\langle x, y \rangle \in X(H) \times X(H) : \Box^{-1}(x) \subseteq y\}$$

we have an LH-space $\langle X(H), \tau_{\mathcal{K}_H}, Q_H \rangle$ on which the dual specialization order is the inclusion relation on $X(H)$.

Given Lax Hilbert algebras $\langle H; \rightarrow, \Box \rangle$, $\langle H'; \rightarrow', \Box' \rangle$ and a semi-homomorphism $h \colon H \rightarrow H'$, we have that $X(h) := \{\langle x', x \rangle \in X(H') \times X(H) : h^{-1}(x') \subseteq x\}$ is an LH-relation, and $X(h)$ is functional if and only if h is an LH-algebra homomorphism.

Conversely, given an LH-space $\langle X, \tau_{\mathcal{K}}, Q \rangle$, the Lax Hilbert algebra $\langle H(X); \rightarrow_{\mathcal{K}}, \Box_Q \rangle$ has the set

$$H(X) := \{U^c : U \in \mathcal{K}\}$$

as universe and operations given, for all $U, V \in H(X)$, as follows:

$$\begin{aligned} U \rightarrow_{\mathcal{K}} V &:= (U \cap V^c]^c = \{x \in X : [x) \cap U \subseteq V\} \\ \Box_Q U &:= \{x \in X : Q(x) \subseteq U\}. \end{aligned}$$

Given LH-spaces $\langle X, \tau_{\mathcal{K}}, Q \rangle$, $\langle X', \tau_{\mathcal{K}'}, Q' \rangle$ and an LH-relation $R \subseteq X \times X'$, we have that the map $H(R) \colon H(X') \rightarrow H(X)$ given by $H(R)(U') := \{x \in X : R(x) \subseteq U'\}$ for all $U \in H(X')$ is a semi-homomorphism of LH-algebras; furthermore, as expected, $H(R)$ is an LH-algebra homomorphism if and only if R is functional.

For every Lax Hilbert algebra $\langle H; \rightarrow, \Box \rangle$, the mapping $\sigma \colon H \rightarrow H(X(H))$ is an isomorphism of Lax Hilbert algebras. Conversely, every LH-space $\langle X, \tau_{\mathcal{K}}, Q \rangle$ is homeomorphic to the space $\langle X(H(X)), \tau_{\mathcal{K}_{H(X)}}, Q_{H(X)} \rangle$ through the map H_X given, for all $x \in X$, by

$$H_X(x) := \{U \in H(X) : x \in U\}.$$

Joining together these observations we have the announced equivalence(s).

Theorem 5.4 ([12], Thm. 2; [11], Thm. 3.13)**.** *The categories* LHSp *and* LHA *are dually equivalent via the functors H and X, and so are the categories* LHSpF *and* LHAH*.*

5.2. **nH-semigroups.** The notion of *filter* of an nH-semigroup $\mathbf{S} = \langle S; \odot, \rightarrow, 0, 1 \rangle$ can be taken to be the same as that of an (implicative) filter of the Hilbert algebra reduct $\langle S; \rightarrow, 1 \rangle$. This is suggested by the observation that any (implicative) filter F of $\langle S; \rightarrow, 1 \rangle$ is closed under the semigroup operation of $\mathbf{S}$. Indeed, assuming $a, b \in F$ we have $\Box a, \Box b \in F$ because F is increasing. Then we can apply Lemma 5.5 below to conclude $a \odot b \in F$.

Lemma 5.5. *Let $\mathbf{S} = \langle S; \odot, \rightarrow, 0, 1 \rangle$ be an nH-semigroup, $F \subseteq S$ a filter and $a, b \in S$. The following are equivalent:*

(i) $a \odot b \in F$.

(ii) $\Box a \in F$ *and* $\Box b \in F$.

Proof. Assume (i) holds, so $a \odot b \in F$. By Definition 3.9.v and the properties of the intuitionistic implication, we have $(a \odot b) \to \Box a = \Box a \to (\Box b \to \Box a) = 1 \in F$. Thus, by $\to$-*modus ponens*, we obtain $\Box a \in F$. A symmetric reasoning allows us to conclude that $\Box b \in F$.

Conversely, assume (ii) holds, so $\Box a, \Box b \in F$. Using again Definition 3.9.v, we have $\Box a \to (\Box b \to (a \odot b)) = ((a \odot b) \to (a \odot b)) = 1 \in F$. Thus, applying $\to$-*modus ponens* twice, we obtain $a \odot b \in F$, as required. □

Let X be a set. We recall that a binary relation R on X is *serial* if $R(x) \neq \emptyset$ for each $x \in X$, and R is *dense* if $R \subseteq R \circ R$.

Proposition 5.6. *Let* $\mathbf{S} = \langle S; \odot, \to, 0, 1\rangle$ *be an nH-semigroup, and let* $\langle X(\mathbf{S}), \tau_{\mathcal{K}_S}, Q_S\rangle$ *be the LH-space dual to the Lax Hilbert algebra reduct* $\langle S; \to, 1\rangle$. *Then,*

(i) $X(\mathbf{S}) \in \mathcal{K}_S$ *(hence,* $\langle X(\mathbf{S}), \tau_{\mathcal{K}_S}\rangle$ *is compact).*

(ii) Q_S *is serial (i.e. for all* $x \in X(\mathbf{S})$*, there is* $y \in X(\mathbf{S})$ *such that* $\langle x, y\rangle \in Q_S$*).*

(iii) $Q_S^{-1}(U \cup V) \in \mathcal{K}_S$ *for all* $U, V \in \mathcal{K}_S$.

Proof. For the first two items, see [11, p. 52 and Thm. 4.5]. To prove (iii), let $U, V \in \mathcal{K}_S$. Then there are $a, b \in S$ such that $U = (\sigma(a))^c$ and $V = (\sigma(b))^c$. We claim that $Q_S^{-1}(U \cup V) = Q_S^{-1}((\sigma(a))^c \cup (\sigma(b))^c) = Q_S^{-1}((\sigma(a) \cap \sigma(b))^c) = (\sigma(a \odot b))^c$. Observe that, on the one hand, we have $x \in Q_S^{-1}((\sigma(a) \cap \sigma(b))^c)$ iff there is $y \in X(\mathbf{S})$ such that $\Box^{-1}(x) \subseteq y$ and either $a \notin y$ or $b \notin y$. On the other hand, by Lemma 5.5, we have $x \in (\sigma(a \odot b))^c$ iff $\Box a \notin x$ or $\Box b \notin x$. The required result then follows from [11, Lemma 3.5], which states that $\Box a \notin x$ iff there is $y \in X(\mathbf{S})$ such that $\Box^{-1}(x) \subseteq y$ and $a \notin y$. □

Definition 5.7. An *nH-space* is an LH-space that further satisfies the three items in Proposition 5.6.

By Definition 5.7, the space dual to (the Lax Hilbert algebra reduct of) every nH-semigroup $\mathbf{S}$ is an nH-space. Conversely, given an nH-space $\langle X, \tau_{\mathcal{K}}, Q\rangle$, the nH-semigroup $\langle H(X); \odot_Q, \to_{\mathcal{K}}, \Box_Q\rangle$ is defined as for LH-spaces, with the extra operation $\odot_Q$ being given, for all $U, V \in H(X)$, by:

$$U \odot_Q V := \Box_Q(U \cap V) = \Box_Q(U) \cap \Box_Q(V) = \{x \in X : Q(x) \subseteq U \cap V\}.$$

Proposition 5.8. *For every nH-space* $\langle X, \tau_{\mathcal{K}}, Q\rangle$*, the algebra* $\langle H(X); \odot_Q, \to_{\mathcal{K}}, \Box_Q\rangle$ *is an nH-semigroup.*

Proof. We know that $\langle H(X); \to_{\mathcal{K}}, \Box_Q\rangle$ is a Lax Hilbert algebra that moreover (by item (i) of Proposition 5.6) is bounded. To complete the proof, it suffices to show that the operation $\odot_Q$ is well defined on $H(X)$ and satisfies the remaining identities of nH-semigroups. Regarding the former claim, let $U, V \in H(X)$, so that $U^c, V^c \in \mathcal{K}$. Then, by the property stated in item (iii) of Proposition 5.6, we have $Q^{-1}(U^c \cup V^c) \in \mathcal{K}$. The result then follows because $(U \odot_Q V)^c = Q^{-1}(U^c \cup V^c)$.

Let us now look at the properties in Definition 3.9. We know from the duality for Lax Hilbert algebras that items (i) and (iii) are satisfied. As for item (ii), commutativity of $\odot_Q$ is clear; associativity easily follows from the observation that $U \odot_Q V = \Box_Q(U) \cap \Box_Q(V)$.

In fact, for all $U, V, W \in H(X)$, we have:

$$
\begin{aligned}
(U \odot_Q V) \odot_Q W &= \Box_Q(U \odot_Q V) \cap \Box_Q(W) \\
&= \Box_Q \Box_Q(U \cap V) \cap \Box_Q(W) \\
&= \Box_Q(U \cap V) \cap \Box_Q(W) \\
&= \Box_Q(U) \cap \Box_Q(V) \cap \Box_Q(W) \\
&= \Box_Q(U) \cap \Box_Q(V \cap W) \\
&= U \odot_Q (V \odot_Q W).
\end{aligned}
$$

For item (iv), we need to show that, for all $U, V \in H(X)$,

$$\{x \in X : Q(x) \subseteq U \cap V\} = \{x \in X : Q(x) \subseteq U \cap (U \cap V^c]^c\}.$$

Let us verify that $U \cap V = U \cap (U \cap V^c]^c$. Let $a \in U \cap V$. As $a \in V$ and V is increasing, $[a) \subseteq V$. So, $[a) \cap U \subseteq V$, i.e, $[a) \cap U \cap V^c = \emptyset$, which entails $a \in (U \cap V^c]^c$. Thus, $U \cap V \subseteq U \cap (U \cap V^c]^c$.

If $a \in U \cap (U \cap V^c]^c$, then $a \in U$ and $[a) \cap U \subseteq V$. Since U is increasing, $[a) \subseteq U$. So, $[a) \cap U = [a) \subseteq V$, and thus $a \in V$. Then we have proved that $U \cap (U \cap V^c]^c \subseteq U \cap V$.

For item (v), observe that $U \to_{\mathcal{K}} (V \to_{\mathcal{K}} W) = (U \cap V) \to_{\mathcal{K}} W$ holds for all $U, V, W \in H(X)$. Then,

$$\Box_Q(U) \to_{\mathcal{K}} (\Box_Q(V) \to_{\mathcal{K}} W) = (\Box_Q(U) \cap \Box_Q(V)) \to_{\mathcal{K}} W = (U \odot_Q V) \to_{\mathcal{K}} W$$

as required. Item (vi) easily follows from the observation that Q is serial. Finally, item (vii) is immediate. □

Proposition 5.9. *For every nH-semigroup $\langle S; \odot, \to, 0, 1\rangle$, the mapping $\sigma \colon S \to H(X(\mathbf{S}))$ is an isomorphism of nH-semigroups.*

Proof. Since we know that σ is a (Lax) Hilbert algebra isomorphism, it suffices to show that σ preserves 0 and the operation $\odot$. Regarding the former, we have $\sigma(0) = \emptyset$ because no proper implicative filter contains 0. Regarding the latter, we need to check that $\sigma(a \odot b) = \sigma(a) \odot_{Q_S} \sigma(b)$ for all $a, b \in S$. Assume $x \in \sigma(a \odot b)$, so $a \odot b \in x$. By Lemma 5.5, this gives us $\Box a, \Box b \in x$. Then, for every $y \in Q_S(x)$, we have $a, b \in y$, as required. On the other hand, assuming $x \notin \sigma(a \odot b)$, we can again apply Lemma 5.5 to conclude that $\Box a \notin x$ or $\Box b \notin x$. Assume the former is the case (a similar reasoning applies if $\Box b \notin x$). Then $x \notin \sigma(\Box a) = \Box_{Q_S} \sigma(a)$. Thus, $Q_S(x) \not\subseteq \sigma(a)$, which means that there is $y \in Q_S(x)$ such that $a \notin y$. Hence, $x \notin \sigma(a) \odot_{Q_S} \sigma(b)$, as required. □

Let nH be the category of nH-semigroups with algebraic homomorphisms, and let nHSp be the category having nH-spaces as objects and *nH-relations* for morphisms, defined as follows.

Definition 5.10. We say that an LH-relation (Definition 5.2) $R \subseteq X \times X'$ between nH-spaces $\langle X, \tau_{\mathcal{K}}, Q\rangle$ and $\langle X', \tau_{\mathcal{K}'}, Q'\rangle$ is an *nH-relation* iff R is functional and serial.

Given nH-spaces $\langle X, \tau_{\mathcal{K}}, Q\rangle$, $\langle X', \tau_{\mathcal{K}'}, Q'\rangle$ and an nH-relation $R \subseteq X \times X'$, we have that the map $H(R) \colon H(X') \to H(X)$ defined as in the case of LH-algebras is an nH-semigroup homomorphism. Indeed, $H(R)$ preserves the bounds (because R is serial) and the $\odot$ operation as well because of Proposition 4.5.

The previous observations entail that every nH-space $\langle X, \tau_{\mathcal{K}}, Q\rangle$ is homeomorphic to the space $\langle X(H(X)), \tau_{\mathcal{K}_{H(X)}}, Q_{H(X)}\rangle$ through the map H_X given as before. Joining together these observations we obtain the announced equivalence.

Theorem 5.11. *The categories* nHSp *and* nH *are dually equivalent via the functors H and X.*

5.3. **Meet semilattices.** The next objective is to present a topological duality for the class of weak implicative semilattices. Since the underlying semilattice of these algebras are not distributive semilattices, we need to appeal to another duality for semilattices that are not necessarily distributive. In this section we will use the duality developed in [13] for bounded semilattices.

Let $\mathbf{S} = \langle S; \wedge, 0, 1\rangle$ be a bounded semilattice. A *filter* is a non-empty set $F \subseteq S$ that is increasing with respect to the semilattice order and closed under finite meets. The set of all filters on $\mathbf{S}$ is denoted by $\mathrm{Fi}(\mathbf{S})$.

As before, a filter P of $\mathbf{S}$ is *irreducible* when, for all $F_1, F_2 \in \mathrm{Fi}(\mathbf{S})$ such that $P = F_1 \cap F_2$, one has $P = F_1$ or $P = F_2$. The set of all irreducible filters on $\mathbf{S}$ is denoted by $X(\mathbf{S})$. A filter P is *prime* when, for all $F_1, F_2 \in \mathrm{Fi}(\mathbf{S})$ such that $F_1 \cap F_2 \subseteq P$, one has $F_1 \subseteq P$ or $F_2 \subseteq P$. Every prime filter is irreducible, while the converse need not hold (cf. Proposition 5.13 below). The following characterizations are quite useful in practice (see [7, 8]). A filter F is irreducible iff for every $a, b \notin F$ there exists $f \in F$ and $c \notin F$ such that $a \wedge f \leq c$ and $b \wedge f \leq c$. A filter P is prime iff, for every $a, b \notin F$, there exists $c \notin F$ such that $a \leq c$ and $b \leq c$.

An *order ideal* of $\mathbf{S}$ is a set $I \subseteq S$ that is decreasing and such that for all $a, b \in I$, there exists $c \in I$ with $a, b \leq c$. It is easy to see that a filter F is irreducible iff $F^c = S - F$ is an order ideal.

The following result shows that every proper filter of a semilattice is the intersection of irreducible filters.

Theorem 5.12 ([7], Thm. 8)**.** *Let $\mathbf{S}$ be a semilattice. Let F be a filter and let I be an order ideal of S such that $F \cap I = \emptyset$. Then there exists $P \in X(\mathbf{S})$ such that $F \subseteq P$ and $P \cap I = \emptyset$.*

There are different generalizations of the notion of distributivity from lattices to semilattices. Here we are going to say that a semilattice $\mathbf{S}$ is *distributive* iff for all $a, b, c \in S$ such that $a \wedge b \leq c$, there exist $a', b' \in S$ such that $a \leq a'$, $b \leq b'$, and $a' \wedge b' = c$ (see [8, 13]). As expected, a bounded lattice $\langle L; \vee, \wedge, 0, 1\rangle$ is distributive (as lattice) if and only if $\langle L, \wedge, 1\rangle$ is a distributive semilattice according to our definition.

Proposition 5.13 ([7])**.** *Let $\mathbf{S} = \langle S; \wedge, 1\rangle$ be a semilattice. The following conditions are equivalent:*

(i) $\mathbf{S}$ *is distributive.*
(ii) $\mathrm{Fi}(\mathbf{S})$ *is a distributive lattice.*
(iii) Every prime filter of $\mathbf{S}$ *is irreducible.*

Given a non-empty set X, consider a family $\mathcal{K} \subseteq \mathcal{P}(X)$ such that $X = \bigcup \mathcal{K}$. In this section we denote by $\tau_{\mathcal{K}}$ the topology on X taking as *subbase* the family $\mathcal{K}$. It is well known that $\tau_{\mathcal{K}}$ consists of $\emptyset$, X, all finite intersections of members of $\mathcal{K}$ and all arbitrary unions of these finite intersections.

Let $\langle X, \tau_{\mathcal{K}}\rangle$ be a topological space. We consider the following collection of subsets of X:

$$\mathrm{S}(X) = \{U \subseteq X : U^c \in \mathcal{K}\}.$$

Let $\mathcal{C}_{\mathcal{K}}(X)$ be the closure system on X generated by $\mathrm{S}(X)$, i.e., $\mathcal{C}_{\mathcal{K}}(X) = \{\bigcap \mathcal{D} : \mathcal{D} \subseteq \mathrm{S}(X)\}$. The closure operator associated to $\mathcal{C}_{\mathcal{K}}(X)$ is denoted by $\mathrm{cl}_{\mathcal{K}}$. The elements of $\mathcal{C}_{\mathcal{K}}(X)$ are closed, but not every closed set is of this form, since $\mathcal{K}$ is only a subbase. Thus, in general

$\mathrm{cl}(Y) \subseteq \mathrm{cl}_{\mathcal{K}}(Y)$ for any $Y \subseteq X$, but $\mathrm{cl}(x) = \mathrm{cl}_{\mathcal{K}}(x)$, for any $x \in X$. The elements of $\mathcal{C}_{\mathcal{K}}(X)$ will be called *subbasic closed* subsets of X. It is clear that $\mathrm{S}(X)$ is closed under finite intersections iff $\mathcal{K}$ is closed under finite unions. Thus, if $\mathcal{K}$ is closed under finite unions and $\emptyset, X \in \mathcal{K}$, then $\mathbf{S}(X) = \langle \mathrm{S}(X), \cap, \emptyset, X\rangle$ is a bounded semilattice, called the *dual semilattice* of $\langle X, \tau_{\mathcal{K}}\rangle$.

Consider a topological space $\langle X, \tau_{\mathcal{K}}\rangle$. From now on we will always assume that the subbase $\mathcal{K}$ is closed under finite unions and that $\emptyset, X \in \mathcal{K}$.

We are now going to define the dual spaces of (bounded) semilattices; the definition we propose does not coincide with the one given in [13], but it is easily seen to be equivalent (see Lemma 3.7 and Prop. 3.8 of [13]).

Definition 5.14. An S-space is a topological space $\langle X, \tau_{\mathcal{K}}\rangle$ satisfying the following:

(S1) $\langle X, \tau_{\mathcal{K}}\rangle$ is a T_0 space.

(S2) $\mathcal{K}$ is a subbase of compact open subsets that is closed under finite unions, and $\emptyset, X \in \mathcal{K}$.

(S3) For each $x \in X$, the set $H_X(x) = \{U \in \mathrm{S}(X) : x \in U\}$ is an irreducible filter of $\mathbf{S}(X)$.

(S4) The map $H_X : X \to X(\mathbf{S}(X))$ is onto.

It is clear that, if $\langle X, \tau_{\mathcal{K}}\rangle$ is a S-space, then $\mathbf{S}(X) = \langle \mathrm{S}(X); \cap, \emptyset, X\rangle$ is a bounded semilattice called the *dual semilattice* of $\langle X, \tau_{\mathcal{K}}\rangle$.

Now we will see how to construct the dual space of a bounded semilattice $\mathbf{S}$. Let $X(\mathbf{S})$ be the poset of all irreducible filters of $\mathbf{S}$. We consider the map $\sigma : S \to \mathrm{Up}(X(\mathbf{S}))$ given $\sigma(a) = \{x \in X(\mathbf{S}) : a \in x\}$, for each $a \in S$. Let $\mathcal{K}_{\mathbf{S}} = \{\sigma(a)^c : a \in S\}$. It is easy to see that $\mathcal{K}_{\mathbf{S}}$ is a subbase for a topology on $X(\mathbf{S})$, closed under finite unions, and $\emptyset, X(\mathbf{S}) \in \mathcal{K}_{\mathbf{S}}$. Then the topological space $\langle X(\mathbf{S}), \tau_{\mathcal{K}_{\mathbf{S}}}\rangle$ is an S-space, and the map $\sigma : \mathbf{S} \to \mathbf{S}(X(\mathbf{S}))$ is an isomorphism of bounded semilattices [13, Prop. 3.10].

Let $\langle X, \tau_{\mathcal{K}}\rangle$ be a S-space and let $\langle X(\mathbf{S}(X)), \tau_{\mathcal{K}_{\mathbf{S}(X)}}\rangle$ be the dual S-space of $\mathbf{S}(X)$. By [13, Prop. 3.12] the map $H_X : X \to X(\mathbf{S}(X))$ is a homeomorphism between the S-spaces. Moreover, $\mathcal{K}_{\mathbf{S}(X)} = \{H_X[U] : U \in \mathcal{K}\}$.

We shall consider the notion of *meet-relation* introduced in [13]. Let $\langle X_1, \tau_{\mathcal{K}_1}\rangle$ and $\langle X_2, \tau_{\mathcal{K}_2}\rangle$ be two S-spaces. A relation $r \subseteq X_1 \times X_2$ is said to be a *meet-relation* if

(R1) For all $U \in \mathrm{S}(X_2)$, $\Box_r(U) = \{x \in X_1 : r(x) \subseteq U\} \in \mathrm{S}(X_1)$.

(R2) For every $x \in X_1$, $r(x) \in \mathcal{C}_{\mathcal{K}_2}(X_2)$.

We say that a meet-relation $r \subseteq X_1 \times X_2$ is serial if

(R3) $r(x) \neq \emptyset$, for each $x \in X_1$.

If $r \subseteq X_1 \times X_2$ is a serial meet-relation, then the map $\Box_r : \mathrm{S}(X_2) \to \mathrm{S}(X_1)$ is a bounded meet-homomorphism. The condition (R3) guarantees that $\Box_r(\emptyset) = \emptyset$.

Definition 5.15 ([8, 13])**.** Let $\langle X_i, \tau_{\mathcal{K}_i}\rangle$, with $i = 1, 2, 3$, be S-spaces, and let $R \subseteq X_1 \times X_2$ and $T \subseteq X_2 \times X_3$ be meet-relations. Let

$$R \circ T = \{(x, z) \in X_1 \times X_3 : \exists y \in X_2\, ((x, y) \in R \text{ and } (y, z) \in T)\}.$$

We define the composition $*$ between T and R as

$$T * R = \{(x, z) \in X_1 \times X_3 : \text{ for every } U \in \mathrm{S}(X_3), (T \circ R)(x) \subseteq U \text{ implies } z \in U\}.$$

Let $h : \mathbf{S}_1 \to \mathbf{S}_2$ be a bounded semilattice homomorphism. Then the relation $r_h \subseteq X(\mathbf{S}_2) \times X(\mathbf{S}_1)$ given by

$$(x, y) \in r_h \text{ iff } h^{-1}(x) \subseteq y,$$

for all $(x, y) \in X(\mathbf{S}_2) \times X(\mathbf{S}_1)$ is a meet-relation.

We consider the category SP whose objects are S-spaces and whose morphisms are the serial meet-relations, where the identity morphism is the dual of the specialization order, and the composition is $*$. Let us denote by BMS the category of bounded semilattices and homomorphisms (that is, maps preserving finite meets and the bounds).

Theorem 5.16 ([13])**.** *The categories* BMS *and* SP *are dually equivalent.*

We next recall the duality for bounded distributive semilattices, which we view as a special case of the preceding duality for semilattices [7, 8]. A topological space $\langle X, \tau \rangle$ is called a *DS-space* if it is sober and the set of all compact open subsets $\mathcal{KO}(X)$ of X is a base for τ. If $\langle X, \tau \rangle$ is a DS-space, then $\langle \mathrm{S}(X); \cap, \emptyset, X \rangle$ is a distributive semilattice. If $\mathbf{D} = \langle D; \wedge, 1 \rangle$ is a distributive semilattice, then the space $\langle X(\mathbf{D}), \tau_{\mathbf{D}} \rangle$, whose the topology is generated by the base $\{\sigma(a)^c : a \in D\}$, is a DS-space. Thus, every DS-space is an S-space [7, 8]. The notion of meet-relation between DS-spaces is the same as for S-spaces. Let BDS be the category whose objects are bounded distributive semilattices and whose morphisms are bounded meet-homomorphisms, and let DSP be the category whose objects are compact DS-spaces and whose morphisms are meet-relations.

Theorem 5.17 ([13])**.** *The categories* BDS *and* DSP *are dually equivalent.*

Recall that, given a poset $\langle X, \leq \rangle$, the algebra $\langle \mathrm{Up}(X); \cap, \to, \emptyset, X \rangle$ is an implicative semilattice where the intuitionistic implication is defined, for all $U, V \in \mathrm{Up}(X)$, by

$$U \to V = \{x \in X : [x) \cap U \subseteq V\}.$$

A spectral duality for implicative semilattices can be obtained by combining the duality for distributive semilattices [7], the duality for Hilbert algebras [5, 6], and the representation results given in [10]. We recall that an implicative space, or IS-space, is a DS-space $\langle X, \tau \rangle$ such that $U \to V := (U \cap V^c]^c \in \mathrm{S}(X)$ where the order $\leq$ is the specialization dual order of $\langle X, \tau \rangle$. The dual implicative semilattice of an implicative space is $\langle \mathrm{S}(X); \cap, \to, X \rangle$. If the IS-space $\langle X, \tau \rangle$ is compact, then $\langle \mathrm{S}(X); \cap, \to, \emptyset, X \rangle$ is a bounded implicative semilattice. Recall from [10] that a *functional meet-relation* is a meet-relation R between two IS-spaces X, Y such that, for every pair $(x, y) \in X \times Y$, if $(x, y) \in R$, then there exists $z \in X$ such that $x \leq z$ and $R(z) = [y)$.

Let IS be the category of compact IS-spaces where the morphisms are the functional meet-relations. By the results of [10, 8], we have that the category of BIM of bounded implicative semilattices whose morphisms are bounded implicative homomorphisms is dually equivalent to the category IS.

Given a semilattice $\mathbf{S} = \langle S; \wedge, 1 \rangle$, we shall say that a map $\Box \colon S \to S$ is a *modal operator* if it satisfies the identities $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box 1 = 1$ (so every nucleus is, in particular, a modal operator: cf. Definition 2.5). The algebra $\langle S; \wedge, \Box, 1 \rangle$ will be called a *modal semilattice*. A $\Box$-*homomorphism* between modal semilattices $\mathbf{S}_1, \mathbf{S}_2$ is a meet-homomorphism $h : S_1 \to S_2$ such that $h(\Box_1 a) = \Box_2 h(a)$, for all $a \in S_1$. Since a modal operator $\Box$ on a semilattice $\mathbf{S}$ is a particular case of meet-homomorphism, we have that the relation $R_\Box \subseteq X(\mathbf{S}) \times X(\mathbf{S})$ given by $(x, y) \in R_\Box$ iff $\Box^{-1}(x) \subseteq y$ is a meet-relation. We are now going to characterize dually the $\Box$-homomorphisms.

Lemma 5.18. *Let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be modal semilattices. Then a meet-homomorphism* $h\colon S_1 \to S_2$ *is a* $\Box$*-homomorphism iff* $R_{\Box_2} \circ r_h = r_h \circ R_{\Box_1}$.

Proof. Let $a \in S_1$ and suppose that $h(\Box_1 a) \nleq \Box_2 h(a)$. Then there exists $x \in X(\mathbf{S}_2)$ such that $h(\Box_1 a) \in x$ and $\Box_2 h(a) \notin x$. As $\Box_2^{-1}(x)$ is a filter, there is $y \in X(\mathbf{S}_2)$ such that $\Box_2^{-1}(x) \subseteq y$ and $h(a) \notin y$. Again, since $h^{-1}(y)$ is a filter of $\mathbf{S}_1$, there exists $z \in X(\mathbf{S}_1)$ such that $h^{-1}(y) \subseteq z$ and $a \notin z$. Then $(x, z) \in R_{\Box_2} \circ r_h$. So, there exists $k \in X(\mathbf{S}_1)$ such that $(x, k) \in r_h$ and $(k, z) \in R_{\Box_1}$. But as $h(\Box_1 a) \in x$, we have $\Box_1 a \in k$, and so $a \in z$, which is a contradiction. Thus $h(\Box_1 a) \leq \Box_2 h(a)$, for all $a \in S_1$. The proof for the other inequality is similar.

We prove the inclusion $R_{\Box_2} \circ r_h \subseteq r_h \circ R_{\Box_1}$. The proof of the other inclusion is similar. Let $x, y \in X(\mathbf{S}_2)$, and $z \in X(\mathbf{S}_1)$ such that $(x, y) \in R_{\Box_2}$ and $(y, z) \in r_h$, i.e., $\Box_2^{-1}(x) \subseteq y$ and $h^{-1}(y) \subseteq z$. Consider the filter $h^{-1}(x)$ of $\mathbf{S}_1$. It is easy to check that the set $\Box_1(z^c) = \{\Box_1 a : a \notin z\}$ is an order ideal. We prove that $h^{-1}(x) \cap \Box_1(z^c) = \emptyset$. Suppose that there exists $a \in h^{-1}(x) \cap \Box_1(z^c)$. Then $h(a) \in x$ and there exists $b \notin z$ such that $a = \Box_1 b$. So, $h(a) = h(\Box_1 b) = \Box_2(h(b)) \in x$, and as $\Box_2^{-1}(x) \subseteq y$, we have $h(b) \in y$. So, $b \in z$, which is a contradiction. Thus, by Theorem 5.12, there exists $w \in X(\mathbf{S}_1)$ such that $h^{-1}(x) \subseteq w$ and $\Box_1^{-1}(w) \subseteq z$, i.e., $(x, z) \in r_h \circ R_{\Box_1}$. $\Box$

Note that the preceding characterization remains valid for distributive semilattices and implicative semilattices with a modal operator.

5.4. **Weak implicative semilattices.** In this section we are going to prove a representation theorem for weak implicative semilattices and subsequently study the corresponding dual spaces.

Lemma 5.19. *Consider a structure* $\langle X, \leq, R\rangle$ *such that* $\langle X, \leq\rangle$ *is a poset and* $R \subseteq X \times X$ *is a relation. The following conditions are equivalent:*

1. $(\leq \circ R) \subseteq (R \circ \leq)$.
2. $\Box_R(U) = \{x \in X : R(x) \subseteq U\} \in \mathrm{Up}(X)$, *for any* $U \in \mathrm{Up}(X)$.
3. $X \to \Box_R(U) = \Box_R(U)$, *for all for all* $U \subseteq X$.

Proof. (1) $\Rightarrow$ (2) Let $U \in \mathrm{Up}(X)$. Let $x, y \in X$ such that $x \leq y$ and $x \in \Box_R(U)$. We prove that $R(y) \subseteq U$. Let $z \in R(y)$. Then $(x, z) \in (\leq \circ R)$. By hypothesis there exists $k \in X$ such that $(x, k) \in R$ and $k \leq z$. As $k \in R(x) \subseteq U$ and $U \in \mathrm{Up}(X)$, we get $z \in U$. Thus, $y \in \Box_R(U)$.

(2) $\Rightarrow$ (3) Let $x \in X$. Then $x \in X \to \Box_R(U)$ iff $[x) \cap X = [x) \subseteq \Box_R(U)$ iff $x \in \Box_R(U)$, for any $U \subseteq X$.

(3) $\Rightarrow$ (1) Let $x, y, z \in X$ such that $x \leq y$ and $(y, z) \in R$. Consider the upset $U = [R(x)) = \{l \in X : \exists k \in R(x)\, (k \leq l)\}$. Then $R(x) \subseteq U$, and thus $x \in \Box_R(U)$. By hypothesis, $X \to \Box_R(U) = \Box_R(U)$. This means that $\Box_R(U)$ is an upset. As $x \leq y$, we get $y \in \Box_R(U)$. So, $R(y) \subseteq U$ and thus $z \in U = [R(x))$. Then there exists $k \in R(x)$ such that $k \leq z$. $\Box$

The preceding characterization gives us one of the ingredients that we need for our definition of weak implicative semilattice (WIS)-frame.

Definition 5.20. A *WIS-frame* is a relational structure $\langle X, \leq, R\rangle$, where $\langle X, \leq\rangle$ is a poset and $R \subseteq X \times X$ is a dense and serial relation on X such that $(\leq \circ R) \subseteq (R \circ \leq)$ and $R \subseteq \leq$.

Lemma 5.21. *Let $\langle X, \leq, R\rangle$ be a WIS-frame. Then $\mathbf{Up}(X) = \langle \mathrm{Up}(X); \cap, \rightharpoonup, \emptyset, X\rangle$ is a weak implicative semilattice where $U \rightharpoonup V := U \to \Box_R(V)$ for all $U, V \in \mathrm{Up}(X)$.*

Proof. We already know that $\langle \mathrm{Up}(X); \cap, \emptyset, X\rangle$ is a bounded semilattice. By definition of $\rightharpoonup$ we get that $U \rightharpoonup V \in \mathrm{Up}(X)$, for any $U, V \in \mathrm{Up}(X)$. By Lemma 5.19 we have that $X \to \Box_R(U) = \Box_R(U)$, and thus $\Box_R(U) \in \mathrm{Up}(X)$, for any $U \in \mathrm{Up}(X)$.

It is easy to check that that the conditions $R \subseteq \leq$ and $R \subseteq R \circ R$, imply that $U \subseteq \Box_R(U)$, and $\Box_R(\Box_R(U)) \subseteq \Box_R(U)$, for all $U \in \mathrm{Up}(X)$, respectively. We prove that $\Box_R(\emptyset) = \emptyset$. If there exists $x \in X$ such that $x \in \Box_R(\emptyset)$, then $R(x) \subseteq \emptyset$, which is a contradiction, because R is serial i.e., $R(x) \neq \emptyset$. Thus, $\Box_R(\emptyset) = \emptyset$.

Let $U, V \in \mathrm{Up}(X)$. We prove that $\Box_R(U \rightharpoonup V) = U \rightharpoonup V$, i.e,

$$\Box_R(U \to \Box_R(V)) = U \to \Box_R(V). \tag{1}$$

We note that

$$\Box_R(U \to \Box_R(V)) \subseteq \Box_R(U) \to \Box_R(\Box_R(V)) \subseteq \Box_R(U) \to \Box_R(V) \subseteq U \to \Box_R(V).$$

For the other inclusion we take $x \in U \to \Box_R(V)$, i.e., $[x) \cap U \subseteq \Box_R(V)$. Let $y \in R(x)$. For each $z \in [y) \cap U$ we get $x \leq y \leq z$ (because $R \subseteq \leq$). Thus $z \in [x) \cap U$, and this implies that $z \in \Box_R(V)$. Therefore we have proved that $U \to \Box_R(V) \subseteq \Box_R(U \to \Box_R(V))$.

The equality (1) entails the remaining properties. Thus, $\mathbf{Up}(X)$ is a weak implicative semilattice as claimed. ☐

The next step is to construct a frame associated with a weak implicative semilattice, and prove that for every weak implicative semilattice $\mathbf{S}$ there exists a frame $\langle X, \leq, R\rangle$ such that $\mathbf{S}$ is isomorphic to a subalgebra of $\mathbf{Up}(X)$. For this we are going to give some technical results that will allow us to obtain the representation Theorem 5.28.

Let $\mathbf{S}$ be a weak implicative semilattice. We recall that $S_\Box := \{a \in S : a = \Box a\}$. It is easy to verify that $S_\Box$ is the universe of a subalgebra $\langle S_\Box; \wedge, \rightharpoonup, 0, 1\rangle$ which is a bounded implicative semilattice (i.e. the operation $\rightharpoonup$ is the relative pseudo-complement on $S_\Box$).

Furthermore, for each weak implicative semilattice $\mathbf{S}$, consider the (irreducible, prime) filters of the corresponding bounded semilattice reduct. As before, the (po)set of all (semilattice) filters on $\mathbf{S}$ is denoted by $\mathrm{Fi}(\mathbf{S})$, while $X(\mathbf{S})$ denotes the (po)set of the irreducible ones. It is easy to see that, for each $F \in \mathrm{Fi}(\mathbf{S})$,

$$\Box^{-1}(F) = \{a : \Box a \in F\} \in \mathrm{Fi}(\mathbf{S}).$$

We define on $\mathrm{Fi}(\mathbf{S})$ a binary relation $R_\Box$ as follows:

$$(F, G) \in R_\Box \text{ iff } \Box^{-1}(F) \subseteq G.$$

Lemma 5.22. *Let $\mathbf{S}$ be a weak implicative semilattice. Then the relation $R_\Box$ on $X(\mathbf{S})$ is serial, dense and, for each $P, Q \in X(\mathbf{S})$, if $(P, Q) \in R_\Box$, then $P \subseteq Q$.*

Proof. As $\Box 0 = 0$, we have that $R_\Box$ is serial. By the inequality $a \leq \Box a$ we have $R_\Box$ is included in the set-theorical relation $\subseteq$. We prove that $R_\Box$ is dense. Let $(P, Q) \in R_\Box$. Consider the subset $(\Box(Q^c)] = \{a \in S : \exists q \notin Q\ (a \leq \Box q)\}$. As Q is irreducible, $(\Box(Q^c)]$ is an order ideal. We prove that $\Box^{-1}(P) \cap (\Box(Q^c)] = \emptyset$. Otherwise, there exists $a \in \Box^{-1}(P)$ and $q \notin Q$ such that $a \leq \Box q$. Then, $\Box a \leq \Box\Box q = \Box q$. So, $\Box q \in P$, and as $(P, Q) \in R_\Box$, we get $q \in Q$, which is a contradiction. Thus, there exists $D \in X(\mathbf{S})$ such that $\Box^{-1}(P) \subseteq D$ and $\Box^{-1}(D) \subseteq Q$, i.e., $R_\Box$ is dense. ☐

Let $\mathbf{S}$ be a weak implicative semilattice with associated implicative semilattice $\mathbf{S}_\Box$. We consider the following families of filters:

$$\mathrm{Fi}^*(\mathbf{S}) = \left\{F \in \mathrm{Fi}(\mathbf{S}) : \Box^{-1}(F) = F\right\}$$

and

$$X^*(\mathbf{S}) = \left\{x \in X(\mathbf{S}) : \Box^{-1}(x) = x\right\} = \mathrm{Fi}^*(\mathbf{S}) \cap X(\mathbf{S}).$$

Proposition 5.23. *For every a weak implicative semilattice* $\mathbf{S}$*, there exists an order isomorphism between the posets* $(\mathrm{Fi}^*(\mathbf{S}), \subseteq)$ *and* $(\mathrm{Fi}(\mathbf{S}_\Box), \subseteq)$*, which restricts to an order isomorphism between* $(X^*(\mathbf{S}), \subseteq)$ *and* $(X(\mathbf{S}_\Box), \subseteq)$*.*

Proof. For each $F \in \mathrm{Fi}^*(\mathbf{S})$, we have that $F \cap S_\Box \in \mathrm{Fi}(\mathbf{S}_\Box)$. And for each $H \in \mathrm{Fi}(\mathbf{S}_\Box)$ it is easy to see that $\Box^{-1}(H) \in \mathrm{Fi}^*(\mathbf{S})$. Moreover it easy to see that $F_1 \subseteq F_2$ iff $F_1 \cap S_\Box \subseteq F_2 \cap S_\Box$, for any $F_1, F_2 \in \mathrm{Fi}^*(\mathbf{S})$. Thus, the map $\alpha : \mathrm{Fi}^*(\mathbf{S}) \to \mathrm{Fi}(\mathbf{S}_\Box)$ given by $\alpha(F) = F \cap S_\Box$ is an order isomorphism.

Let $x \in X^*(\mathbf{S})$. We prove that $x \cap S_\Box$ is an irreducible filter of $S_\Box$. Let $a, b \in S_\Box$ such that $a, b \notin x$. As x is irreducible, there exist $c \in x$ and there exists $d \notin x$ such that $a \wedge c \leq d$ and $b \wedge c \leq d$. Then $a \leq c \rightharpoonup d$ and $b \leq c \rightharpoonup d$. If $c \rightharpoonup d \in x$, then $c \wedge (c \rightharpoonup d) \leq \Box d \in x$, but as $\Box^{-1}(x) = x$, we get that $d \in x$, which is impossible. So, $c \rightharpoonup d \notin x$, and thus $x \cap S_\Box$ is prime. We recall that in implicative semilattices the notions of prime filter and irreducible filter coincide.

Let $x \in X(\mathbf{S}_\Box)$. We prove that $\Box^{-1}(x)$ is an irreducible filter of S. Let $a, b \notin x$. Then there exists $d \notin x$ such that $a \leq d$ and $b \leq d$. As $d = \Box d \in S_\Box$, we get $d \notin \Box^{-1}(x)$. If we take $c = 1$, we have that $a \wedge 1 \leq d$ and $b \wedge 1 \leq d$, and thus $\Box^{-1}(x)$ is an irreducible filter of S. □

Let $\mathbf{S}$ be a semilattice. For each $D \subseteq S$ the filter generated by a subset D of S is denoted by $\mathrm{Fg}(D)$.

Lemma 5.24. *Let* $\mathbf{S}$ *be a weak implicative semilattice. Let* $F \in \mathrm{Fi}(\mathbf{S})$ *and* $y \in X^*(\mathbf{S})$ *such that* $\Box^{-1}(F) \subseteq y$*. Then there exists* $z \in X(\mathbf{S})$ *such that* $F \subseteq z$ *and* $\Box^{-1}(z) \subseteq y$*.*

Proof. Consider the family $\mathcal{F} = \left\{H \in \mathrm{Fi}(\mathbf{S}) : F \subseteq H \text{ and } \Box^{-1}(H) \subseteq y\right\}$. We note that $\mathcal{F} \neq \emptyset$, because $F \in \mathcal{F}$. Then by Zorn's lemma, there exists a maximal element z in $\mathcal{F}$. We prove that z is irreducible. Let $a, b \notin z$. Consider the filters $z_a = \mathrm{Fg}\left(z \cup \{a\}\right)$ and $z_b = \mathrm{Fg}\left(z \cup \{b\}\right)$. Then $z_a, z_b \notin \mathcal{F}$. So, $\Box^{-1}(z_a) \nsubseteq y$ and $\Box^{-1}(z_b) \nsubseteq y$. Then there are $c, d \in S$ such that $\Box c \in z_a$, $\Box d \in z_b$, and $c, d \notin y$. By the description of the filter generated by a subset we can assert that there exists an element $l \in z$ such that $l \wedge a \leq \Box c$ and $l \wedge b \leq \Box d$. As y is irreducible, there are $q \in y$ and $u \notin y$ such that $c \wedge q \leq u$ and $d \wedge q \leq u$. Then $\Box(c \wedge q) = \Box c \wedge \Box q \leq \Box u$ and $\Box(d \wedge q) = \Box d \wedge \Box q \leq \Box u$. So, $\Box c \leq \Box q \rightharpoonup \Box u = q \rightharpoonup u$ and $\Box d \leq \Box q \rightharpoonup \Box u = q \rightharpoonup u$. Then

$$l \wedge a \leq \Box c \leq q \rightharpoonup u \text{ and } l \wedge b \leq \Box d \leq q \rightharpoonup u.$$

We note that $q \rightharpoonup u \notin z$. Otherwise, as $q \in y$, $q \rightharpoonup u = \Box(q \rightharpoonup u) \in z$ and $\Box^{-1}(z) \subseteq y$, we get $q \wedge (q \to u) \in y$. So, $\Box u \in y$, and since $y = \Box^{-1}(y) \in X^*(\mathbf{S})$, we have that $u \in y$, which is a contradiction. Therefore we have found an element $l \in z$, and an element $q \rightharpoonup u \notin z$ such that $l \wedge a \leq q \rightharpoonup u$ and $l \wedge b \leq q \rightharpoonup u$. Thus, z is irreducible. □

Lemma 5.25. *Let* $\mathbf{S}$ *be a weak implicative semilattice and* $F \in \mathrm{Fi}(\mathbf{S})$*. Let* $a \in S$ *such that* $\Box a \notin F$*. Then there exist* $x \in X(\mathbf{S})$ *and* $y \in X^*(\mathbf{S})$ *such that* $F \subseteq x$*,* $(x, y) \in R_\Box$ *and* $a \notin y$*.*

Proof. Let $\Box a \notin F$. Consider the filter $F \cap S_\Box$. Then $\Box a \notin F \cap S_\Box$, and thus there exists an irreducible filter D of $S_\Box$ such that $F \cap S_\Box \subseteq D$ and $\Box a \notin D$. So, $\Box^{-1}(F) \subseteq \Box^{-1}(D) = y$ and $a \notin y$. As $\Box^{-1}(F) \subseteq y$ by Lemma 5.24 there exists $x \in X(\mathbf{S})$ such that $F \subseteq x$ and $\Box^{-1}(x) \subseteq y$, i.e., $(x, y) \in R_\Box$. ☐

Lemma 5.26. *Let* $\mathbf{S}$ *be a weak implicative semilattice. Let* $F \in \mathrm{Fi}(\mathbf{S})$. *Then* $a \rightharpoonup b \notin F$ *iff there exists* $y \in X(\mathbf{S})$ *and* $z \in X^*(\mathbf{S})$ *such that* $F \subseteq y$, $(y, z) \in R_\Box$, $a \in y$, *and* $b \notin z$.

Proof. Assume that $a \rightharpoonup b \notin F$. We note that $\mathrm{Fg}\,(F \cup \{a\}) \cap (\Box b] = \emptyset$, because if there exists $f \in F$ such that $f \wedge a \leq \Box b$, then $f \leq a \rightharpoonup \Box b = a \rightharpoonup b \in F$, which is impossible. Thus there exists $x \in X(\mathbf{S})$ such that $F \subseteq x$, $a \in x$ and $\Box b \notin x$. By Lemma 5.25, there exists $y \in X(\mathbf{S})$ and $z \in X^*(\mathbf{S})$ such that $x \subseteq y$, $(y, z) \in R_\Box$, and $b \notin z$.

Suppose that there exists $y \in X(\mathbf{S})$ and $z \in X^*(\mathbf{S})$ such that $F \subseteq y$, $(y, z) \in R_\Box$, $a \in y$, and $b \notin z$. If $a \rightharpoonup b \in F$, then $a \wedge (a \rightharpoonup b) \in y$, and so $\Box b \in y$. As $(y, z) \in R_\Box$ we get $b \in z$, which is a contradiction. ☐

Let $\mathbf{S}$ be a weak implicative semilattice. The filter and the downset generated by a subset D of $S_\Box$ are denoted $\mathrm{Fg}(D)_{S_\Box}$ and $(D]_{S_\Box}$, respectively.

Lemma 5.27. *Let* $\mathbf{S}$ *be a weak implicative semilattice. Let* $F \in \mathrm{Fi}(\mathbf{S})$ *and let* $G \subseteq S_\Box$ *such that* $(G]_{S_\Box}$ *is an order ideal of* $S_\Box$. *If* $\mathrm{Fg}(F \cap S_\Box)_{S_\Box} \cap (G]_{S_\Box} = \emptyset$, *then there exists* $x \in X^*(\mathbf{S})$ *such that* $\Box^{-1}(F) \subseteq x$ *and* $x \cap G = \emptyset$.

Proof. Since $\mathrm{Fg}(F \cap S_\Box)_{S_\Box} \cap (G]_{S_\Box} = \emptyset$ and $S_\Box$ is an implicative semilattice, there exists a prime filter y of $S_\Box$ such that $\mathrm{Fg}(F \cap S_\Box)_{S_\Box} \subseteq y$ and $y \cap (G]_{S_\Box} = \emptyset$. Then $F \cap S_\Box \subseteq y$ and $\Box^{-1}(y) \cap G = \emptyset$. So, $\Box^{-1}(F) \subseteq \Box^{-1}(y)$. By Proposition 5.23 $\Box^{-1}(y) = x \in X^*(\mathbf{S})$. Thus, $\Box^{-1}(F) \subseteq x$ and $x \cap G = \emptyset$. ☐

Let $\mathbf{S}$ be a weak implicative semilattice. We define a map $\sigma : S \to \mathrm{Up}(X(\mathbf{S}))$ as

$$\sigma(a) = \{x \in X(\mathbf{S}) : a \in x\},$$

for each $a \in S$. It is clear that σ is a bounded semilattice homomorphism, i.e., $\sigma(0) = \emptyset$, $\sigma(1) = X(S)$, and $\sigma(a \wedge b) = \sigma(a) \wedge \sigma(b)$, for every $a, b \in S$.

Theorem 5.28 (Representation theorem)**.** *Let* $\mathbf{S}$ *be a weak implicative semilattice. Then the structure* $\langle X(\mathbf{S}), \subseteq, R_\Box \rangle$ *is a WIS-frame, and the map* $\sigma : \mathbf{S} \to \mathrm{Up}(X(\mathbf{S}))$ *is an injective weak implicative semilattice homomorphism, i.e.,* $\sigma(a \rightharpoonup b) = \sigma(a) \rightharpoonup \sigma(b)$, *for all* $a, b \in S$.

Proof. By Theorem 5.12, σ is an order-isomorphism, i.e., $a \leq b$ iff $\sigma(a) \subseteq \sigma(b)$, for every $a, b \in S$. Thus, σ is injective. It is easy to see that $(\subseteq \circ R_\Box) \subseteq (R_\Box \circ \subseteq)$. By Lemma 5.22 we have that $\langle X(\mathbf{S}), \subseteq, R_\Box \rangle$ is a WIS-frame. Let $a, b \in S$. We prove that $\sigma(a \rightharpoonup b) = \sigma(a) \rightharpoonup \sigma(b)$. Let $x \in X(\mathbf{S})$ and we suppose that $a \rightharpoonup b \notin x$. Then by Lemma 5.26 there exist $y, z \in X(S)$ such that $x \subseteq y$, $(y, z) \in R_\Box$, $a \in y$ and $b \notin z$. So $y \in [x) \cap \sigma(a)$ but $y \notin \Box_{R_\Box}(\sigma(a))$. Thus, $x \notin \sigma(a) \rightharpoonup \sigma(b)$. So, we have shown that $\sigma(a) \rightharpoonup \sigma(b) \subseteq \sigma(a \rightharpoonup b)$.

Suppose $a \rightharpoonup b \in x$, and let $y, z \in X(S)$ be such that $x \subseteq y$, $(y, z) \in R_\Box$, and $a \in y$. Then $a \wedge (a \rightharpoonup b) \leq \Box b \in y$, and as $(y, z) \in R_\Box$ we have $b \in z$. Then, $\sigma(a \rightharpoonup b) \subseteq \sigma(a) \rightharpoonup \sigma(b)$.

Thus, $\mathbf{S}$ is isomorphic to a subalgebra of $\mathbf{Up}(X(\mathbf{S})) = \langle \mathrm{Up}(X(\mathbf{S})); \cap, \cup, \rightharpoonup, \emptyset, X \rangle$. ☐

Our next aim is to identify the S-spaces (Definition 5.14) that correspond to weak implicative semilattices. Theorem 5.28, together with the duality for bounded semilattices, motivates the following definition.

Definition 5.29. An *implicative S-space* is a triple $\langle X, R, \tau_{\mathcal{K}}\rangle$ such that $\langle X, \tau_{\mathcal{K}}\rangle$ is an S-space, and $R \subseteq X \times X$ is a relation such that:

(IS1) R is dense, serial and $R \subseteq \leq$, where $\leq$ is the specialization dual order of $\langle X, \tau_{\mathcal{K}}\rangle$.
(IS2) $R(x) \in \mathcal{C}_{\mathcal{K}}(X)$, for each $x \in X$.
(IS3) For $U, V \in \mathrm{S}(X)$, $U \rightharpoonup V = \{x \in X : [x) \cap U \subseteq \Box_R(V)\} \in \mathrm{S}(X)$.
(IS4) $(\leq \circ R) \subseteq (R \circ \leq)$.

Remark 5.30. If $\langle X, R, \tau_{\mathcal{K}}\rangle$ is an implicative S-space, then $\langle X, \leq, R\rangle$ is a WIS-frame (see Definition 5.20) and $\langle \mathrm{S}(X), \cap, \rightharpoonup, \emptyset, X\rangle$ is subalgebra of the weak implicative semilattice $\mathbf{Up}(X) = \langle \mathrm{Up}(X); \cap, \rightharpoonup, \emptyset, X\rangle$.

Proposition 5.31. *Let* $\mathbf{S}$ *be a weak implicative semilattice. Then* $\langle X(\mathbf{S}), R_{\Box}, \tau_{\mathbf{S}}\rangle$ *is an implicative S-space, and the map* $\sigma : \mathbf{S} \to \mathrm{S}(X(\mathbf{S}))$ *is an isomorphism of weak implicative semilattices.*

Proof. The result easily follows from the duality for bounded semilattices, Lemma 5.19, and Theorem 5.28. □

Proposition 5.32. *Let* $\langle X, R, \tau_{\mathcal{K}}\rangle$ *be an implicative S-space. Then the map* $H_X : X \to X(\mathrm{S}(X))$ *is a homeomorphism between the implicative S-spaces* $\langle X, R, \tau_{\mathcal{K}}\rangle$ *and* $\langle X(\mathrm{S}(X)), R_{\Box_R}, \tau_{\mathcal{K}_{\mathrm{S}(X)}}\rangle$ *such that,*

$$(x, y) \in R \text{ iff } (H_X(x), H_X(y)) \in R_{\Box_R},$$

for every $x, y \in X$.

Proof. By the duality for bounded semilattices it is clear that the map H_X is a homeomorphism. Let $x, y \in X$. If $(x, y) \in R$, then it is easy to see that $\Box_R^{-1}(H_X(x)) \subseteq H_X(y)$, i.e., $(H_X(x), H_X(y)) \in R_{\Box_R}$. Suppose that $(x, y) \notin R$. Then $y \notin R(x)$ and as $R(x) \in \mathcal{C}_{\mathcal{K}}(X)$, then there exists $U \in \mathrm{S}(X)$ such that $R(x) \subseteq U$ and $y \notin U$. So, $\Box_R(U) \in H_X(x)$ and $U \notin H_X(y)$. Thus, $(H_X(x), H_X(y)) \notin R_{\Box_R}$. □

As shown in [13, Thm. 4.8], the compact DS-spaces are precisely the compact S-spaces $\langle X, \tau_{\mathcal{K}}\rangle$ such that $\mathcal{K}$ is a base for $\tau_{\mathcal{K}}$. It is possible to prove that, under these conditions, $\mathcal{K}$ is the set of *all* compact and open subsets of $\langle X, \tau_{\mathcal{K}}\rangle$. This fact is used in the next result.

Lemma 5.33. *Let* $\langle X, R, \tau_{\mathcal{K}}\rangle$ *be an implicative S-space. Let* $X^* = \{x \in X : (x, x) \in R\}$. *Then the subspace* $\langle X^*, \tau_{X^*}\rangle$*, where* $\tau_{X^*} = \{U \cap X^* : U \in \tau_{\mathcal{K}}\}$*, is a compact DS-space.*

Proof. Let $\langle X, R, \tau_{\mathcal{K}}\rangle$ be an implicative S-space. By duality we can assert that there exists a weak implicative semilattice $\mathbf{S}$ such that $X = X(\mathbf{S})$. By results on general topology, $\mathcal{K}^* = \{U \cap X^* : U \in \mathcal{K}\}$ is a subbase of τ_{X^*}.

We prove that $\mathcal{K}^*$ is base for τ_{X^*}. Let $U, V \in \mathcal{K}$. We need to prove that $U \cap V \cap X^*$ is union of elements of $\mathcal{K}^*$. Then there exist $a, b \in S$ such that $U = \sigma(a)^c$ and $V = \sigma(b)^c$. If $x \in U \cap V \cap X^* = \sigma(a)^c \cap \sigma(b)^c \cap X^*$, then $a, b \notin x$ and as $(x, x) \in R$, we get $\Box a, \Box b \notin x$. So, $\Box a, \Box b \notin x \cap S_{\Box}$ and by Proposition 5.23, the set $x \cap S_{\Box}$ is an irreducible filter of the implicative semilattice $\mathbf{S}_{\Box}$. By Proposition 5.13, $x \cap S_{\Box}$ is prime. Then there exists $c \in S_{\Box}$ such that $c \notin x \cap S_{\Box}$, $\Box a \leq c$ and $\Box b \leq c$. Taking into account that $\sigma(\Box d)^c \cap X^* = \sigma(d)^c \cap X^*$, for any $d \in S$, we have that $\sigma(c)^c \cap X^* \subseteq \sigma(\Box a)^c \cap X^* = \sigma(a)^c \cap X^*$ and $\sigma(c)^c \cap X^* \subseteq \sigma(\Box b)^c \cap X^* = \sigma(b)^c \cap X^*$. Thus, $\mathcal{K}^*$ is a base for τ_{X^*}. Since $X \in \mathcal{K}$, then $X^* \in \mathcal{K}^*$. Then $\langle X^*, \tau_{X^*}\rangle$ is a compact S-space where $\mathcal{K}^*$ is a base for τ_{X^*}. Thus, $\langle X^*, \tau_{X^*}\rangle$ is a compact DS-space, i.e., is the space of a bounded distributive semilattice [7, 8].

We prove that $R = \leq$ on X^*. Always, $R \subseteq \leq$. Let $x, y \in X^*$ such that $x \leq y$. Suppose that $y \notin R(x)$. By condition (IS2) of Definition 5.29 there exists $U \in \mathrm{S}(X)$ such that $R(x) \subseteq U$ and $y \notin U$. Then $x \in \Box_R(U)$. As $(y, y) \in R$ and $y \notin U$ we have $y \notin \Box_R(U)$, which is a contradiction, because $\Box_R(U) \in \mathrm{Up}(X)$. As a consequence of this fact we get that, $U \Rightarrow V = U \to V$, for every $U, V \in \mathrm{S}(X^*) = \{U \subseteq X^* : (X^* - U) \in \mathcal{K}^*\}$. Thus, $\langle \mathrm{S}(X^*), \to \cap, \to, \emptyset, X^* \rangle$ is a bounded implicative semilattice. □

Our next aim is to extend the representation of weak implicative semilattices through implicative S-spaces to a full categorical duality. To this end, we need to specify which are the morphisms between two objects in the respective categories.

Proposition 5.34. *Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be weak implicative semilattices, and let $h : S_1 \to S_2$ be a bounded meet homomorphism. Then the following conditions are equivalent:*

1. $h(\Box_1 a) \leq \Box_2 h(a)$, *for all* $a \in S_1$.
2. $R_{\Box_2} \circ r_h \subseteq r_h \circ R_{\Box_1}$.

Proof. (1) ⇒ (2) Let $x, y \in X(\mathbf{S}_2)$, and $z \in X(\mathbf{S}_1)$ such that $(x, y) \in R_{\Box_2}$ and $h^{-1}(y) \subseteq z$. We prove that $\Box_1^{-1}\left(h^{-1}(x)\right) \subseteq z$. Let $a \in \Box_1^{-1}\left(h^{-1}(x)\right)$. Then, $\Box_1 a \in h^{-1}(x)$, i.e, $h(\Box_1 a) \in x$. Since $h(\Box_1 a) \leq \Box_2 h(a)$, we get $h(a) \in \Box_2^{-1}(x)$. So, $h(a) \in y$ and thus $a \in z$. By Lemma 5.24 there exists $w \in X(\mathbf{S}_1)$ such that $h^{-1}(x) \subseteq w$ and $(w, z) \in R_{\Box_1}$.

(2) ⇒ (1) Suppose that there exists $a \in S_1$ such that $h(\Box_1 a) \nleq \Box_2 h(a)$. Then there exist $x \in X(\mathbf{S}_2)$ such that $h(\Box_1 a) \in x$ and $\Box_2(h(a)) \notin x$. Then there exists $y \in X(\mathbf{S}_2)$ such that $(x, y) \in R_2$ and $h(a) \notin y$. Again, there exists $z \in X(\mathbf{S}_1)$ such that $h^{-1}(y) \subseteq z$, and $a \notin z$. By hypothesis, there is $w \in X(\mathbf{S}_1)$ such that $h^{-1}(x) \subseteq w$ and $(w, z) \in R_{\Box_1}$. As $h(\Box_1 a) \in x$, $\Box_1 a \in w$, and since $(w, z) \in R_{\Box_1}$, we have that $a \in z$, which is impossible. Thus, $h(\Box_1 a) \leq \Box_2 h(a)$, for all $a \in S_1$. □

In the following result we need to recall the Definition 4.8 of semi-homomorphism between weak implicative semilattices.

Proposition 5.35. *Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be weak implicative semilattices. Let $h : S_1 \to S_2$ be a bounded semi-homomorphism. Then the following conditions are equivalent:*

1. $h(a) \rightharpoonup_2 h(b) \leq h(a \rightharpoonup_1 b)$, *for all* $a, b \in S_1$.
2. *For all $x \in X(\mathbf{S}_2)$, $y \in X(\mathbf{S}_1)$, and $z \in X^*(\mathbf{S}_1)$, if $h^{-1}(x) \subseteq y$ and $(y, z) \in R_{\Box_1}$, then there exists $w \in X^*(\mathbf{S}_2)$ such that $(x, w) \in R_{\Box_2}$ and $z = h^{-1}(w)$.*

Proof. (1) ⇒ (2) Let $x \in X(\mathbf{S}_2)$, $y \in X(\mathbf{S}_1)$, and $z \in X^*(\mathbf{S}_1)$ such that $h^{-1}(x) \subseteq y$ and $(y, z) \in R_{\Box_1}$. We prove that the subset

$$\left(h(z^c) \cap (S_2)_{\Box_2}\right]_{(S_2)_{\Box_2}} = \left\{a \in S_2 : \exists c \notin z \left(a \leq h(c) \in (S_2)_{\Box_2}\right)\right\}$$

is an order ideal of $\mathbf{S}_{\Box_2}$. Let $a, b \in \left(h(z^c) \cap (S_2)_{\Box_2}\right]_{(S_2)_{\Box_2}}$. Then there exist $c_1, c_2 \notin z$ such that $h(c_1)$,$h(c_2) \in (S_2)_{\Box_2}$, $a \leq h(c_1)$ and $b \leq h(c_2)$. As z is an irreducible filter of $\mathbf{S}_1$, there are $d \in z$ and $c \notin z$ such that $c_1 \wedge d \leq c$ and $c_2 \wedge d \leq c$. Then $c_1 \leq d \rightharpoonup_1 c$ and $c_2 \leq d \rightharpoonup_1 c$. We note that $d \rightharpoonup_1 c_1 \notin z$, because otherwise we would get $d \wedge (d \rightharpoonup_1 c) \leq \Box_1 c \in z$, and as $z = \Box_1^{-1}(z)$, we have $c \in z$, which is impossible. Then, $a \leq h(c_1) \leq h(d \rightharpoonup_1 c)$ and $b \leq h(c_2) \leq h(d \rightharpoonup_1 c)$. As

$$\Box_2 h(c \rightharpoonup_1 c) = 1 \rightharpoonup_2 h(d \rightharpoonup_1 c) = h(1) \rightharpoonup_2 h(d \rightharpoonup_1 c) \leq h(1 \rightharpoonup_1 (d \rightharpoonup_1 c)) = h(d \rightharpoonup_1 c)$$

we have that $h(d \rightharpoonup_1 c) \in (S_2)_{\Box_2}$. Therefore we have proved that $\big(h(z^c) \cap (S_2)_{\Box_2}\big]_{(S_2)_{\Box_2}}$ is an order ideal of $\mathbf{S}_{\Box_2}$.

We consider in $\mathbf{S}_{\Box_2}$ the filter $\mathrm{Fg}(Z) \cap (S_2)_{\Box_2}$ where $Z = \Box_2^{-1}(x) \cup h(z)$. We prove that

(2) $$\mathrm{Fg}(Z) \cap (S_2)_{\Box_2} \cap \big(h(z^c) \cap (S_2)_{\Box_2}\big]_{(S_2)_{\Box_2}} = \emptyset.$$

Assume otherwise. Then there exists $f \in \mathrm{Fg}(Z) \cap (S_2)_{\Box_2}$ and $c \notin z$ such that $f \leq h(c)$ and $\Box_2 h(c) = h(c)$. So, there exist $a \in \Box_2^{-1}(x)$ and $b \in z$ such that $a \wedge h(b) \leq f \leq h(c)$. So, $a \leq h(b) \rightharpoonup_2 h(c) \leq h(b \rightharpoonup_1 c)$. Then $\Box_2 a \leq \Box_2 (h(b) \rightharpoonup_2 h(c)) = h(b) \rightharpoonup_2 h(c) \in x$. Since $h(b) \rightharpoonup_2 h(c) \leq h(b \rightharpoonup_1 c) \in x$, we have $b \rightharpoonup_1 c \in y$, because $h^{-1}(x) \subseteq y$. As $(y,z) \in R_{\Box_1}$, we get $y \subseteq z$. So, $b \wedge (b \rightharpoonup_1 c) \leq \Box_1 c \in z$. But as $z = \Box_1^{-1}(z)$, we have $c \in z$, which is an absurd. Then (2) is valid.

Thus by Lemma 5.27 there exists $w \in X^*(\mathbf{S}_2)$ such that $\mathrm{Fg}(Z) \subseteq w$ and $w \cap h(z^c) \cap (S_2)_{\Box_2} = \emptyset$. As $Z \subseteq \mathrm{Fg}(Z) \subseteq w$, we have $\Box^{-1}(x) \subseteq w$ and $h(z) \subseteq w$. So, $z \subseteq h^{-1}(w)$.

We prove that $h^{-1}(w) \subseteq z$. Let $a \in h^{-1}(w)$. Then $h(a) \in w$, and as $\Box_2^{-1}(w) = w$, we get

(3) $$\Box_2 h(a) \in w.$$

Suppose that $a \notin z$. As $\Box_1^{-1}(z) = z$, $\Box_1 a \notin z$. Then

(4) $$h(\Box_1 a) \in h(z^c).$$

On the other hand, we note that

$$\Box_2 h(\Box_1 a) = 1 \rightharpoonup_2 h(\Box_1 a) = h(1) \rightharpoonup_2 h(\Box_1 a) \leq h(1 \rightharpoonup_1 \Box_1 a) = h(\Box_1 \Box_1 a) = h(\Box_1 a).$$

As $h(\Box_1 a) \leq \Box_2 h(\Box_1 a)$, we get that $\Box_2 h(\Box_1 a) = h(\Box_1 a)$, i.e.,

(5) $$h(\Box_1 a) \in (S_2)_{\Box_2}.$$

Then by (3), (4), and (5), we have that $h(\Box_1 a) \in w \cap h(z^c) \cap (S_2)_{\Box_2}$, which is a contradiction. Therefore we have found a filter $w \in X^*(\mathbf{S}_2)$ such that, $\Box^{-1}(x) \subseteq w$ and $z = h^{-1}(w)$.

(2) $\Rightarrow$ (1) Suppose that there exist $a, b \in S_1$ such that $h(a) \rightharpoonup_2 h(b) \nleq h(a \rightharpoonup_1 b)$. Then there exists $x \in X(\mathbf{S}_2)$ such that $h(a) \rightharpoonup_2 h(b) \in x$ and $h(a \rightharpoonup_1 b) \notin x$, i.e., $a \rightharpoonup_1 b \notin h^{-1}(x)$. As h is a meet-homomorphism, $h^{-1}(x)$ is a filter of $\mathbf{S}_1$. Then by Lemma 5.26 there are $y \in X(\mathbf{S}_1)$ and $z \in X^*(\mathbf{S}_1)$ such that $h^{-1}(x) \subseteq y$, $a \in y$, $(y,z) \in R_{\Box_1}$, and $b \notin z$. By hypothesis, there exists $w \in X^*(\mathbf{S}_2)$ such that $(x,w) \in R_{\Box_2}$ and $z = h^{-1}(w)$. As $h(a) \rightharpoonup_2 h(b) \in x$, and $(x,w) \in R_{\Box_2}$, so we have that $h(a) \rightharpoonup_2 h(b) \in w$. Since $a \in y \subseteq z$, we have $h(a) \in w$. Then, $(h(a) \rightharpoonup_2 h(b)) \wedge h(b) \in w$, and this implies $\Box_2 h(b) \in w$, but as $w \in X^*(\mathbf{S}_2)$, $h(b) \in w$. So, $b \in h^{-1}(w) = z$, which is impossible. Thus, $h(a) \rightharpoonup_2 h(b) \leq h(a \rightharpoonup_1 b)$, for all $a, b \in S_1$. □

Taking into account Propositions 5.35 and 5.34, we can consider two notions of morphism between weak implicative semilattices. It is easy to see that the composition of (semi-) homomorphisms is a (semi-) homomorphism. We can therefore consider two categories:

		Objects		Arrows
WISS	=	weak implicative semilattices	+	semi-homomorphisms
WISH	=	weak implicative semilattices	+	homomorphisms.

Remark 5.36. We recall that if $h : \mathbf{S}_1 \to \mathbf{S}_2$ is a bounded meet-homomorphism then the relation $r_h \subseteq X(\mathbf{S}_2) \times X(\mathbf{S}_1)$ given by $(x,y) \in r_h$ iff $h^{-1}(x) \subseteq y$, is a meet-relation. Given $z \in X(\mathbf{S}_1)$ the following conditions are equivalent:

1. There exists $w \in X(\mathbf{S}_2)$ such that $z = h^{-1}(w)$.

2. There exists $w \in X(\mathbf{S}_2)$ such that $r_h(w) = [z)$.

Indeed. We note that if there exists $w \in X(\mathbf{S}_2)$ such that $z = h^{-1}(w)$, then $y \in r_h(w)$ iff $h^{-1}(w) \subseteq y$ iff $z \subseteq y$. Thus, $r_h(w) = [z)$. If there is $w \in X(\mathbf{S}_2)$ such that $r_h(w) = [z)$, then it is clear that $h^{-1}(w) \subseteq z$. If there exists $a \in z$ such that $a \notin h^{-1}(w)$, then there exists $y \in X(\mathbf{S}_1)$ such that $h^{-1}(w) \subseteq y$ and $a \notin y$. But this implies that $y \in r_h(w) = [z)$, i.e., $z \subseteq y$. Then $a \in y$, which is a contradiction. Thus $z = h^{-1}(w)$.

Let $\langle X_1, R_1, \tau_{\mathcal{K}_1}\rangle$ and $\langle X_2, R_2, \tau_{\mathcal{K}_2}\rangle$ be implicative S-spaces. Let $X_1^* = \{x \in X_1 : (x,x) \in R_1\}$ and $X_2^* = \{x \in X_2 : (x,x) \in R_2\}$. Let $r \subseteq X_1 \times X_2$ be a meet-relation. We consider the following conditions:

(R1) If $(x,y) \in R_1$ and $(y,z) \in r$, then there exists $w \in X_2$ such that $(x,w) \in r$ and $(w,z) \in R_2$.

(R2) If $(x,y) \in r$, $(y,z) \in R_2$ and $z \in X_2^*$, then there exists $w \in X_1^*$ such that $(x,w) \in R_1$ and $r(w) = [z)$.

Propositions 5.35 and 5.34, give us the following corollary.

Corollary 5.37. *Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be weak implicative semilattices, and let $h : S_1 \to S_2$ be a bounded meet-homomorphism. Consider the meet-relation $r_h \subseteq X(\mathbf{S}_2) \times X(\mathbf{S}_1)$ defined by $(x,y) \in r_h$ iff $h^{-1}(x) \subseteq y$. Then:*

(i) h is a semi-homomorphism iff r_h satisfies (R1).

(ii) h is a homomorphism iff r_h satisfies (R1) *and* (R2).

Let ISp be the class of implicative S-spaces. It is easy to see that the composition of meet-relations satisfying (R1) or (R2) is a meet-relation satisfying (R1) or (R2), respectively. Thus, we can define the following two categories with the same objects.

		Objects		Arrows
ISpS	=	ISp	+	meet-relations satisfying (R1)
ISpH	=	ISp	+	meet-relations satisfying (R1) and (R2).

By the categorical duality for semilattices given in [13], together with Propositions 5.31, 5.32, 5.35 and 5.34, we have the following result.

Theorem 5.38. *The categories* WISS *(resp.* WISH*) and* ISpS *(resp.* ISpH*) are dually equivalent.*

5.5. **$\oplus$-implicative semilattices.** To conclude the section, we shall extend the duality for bounded implicative semilattices to the case of the $\oplus$-implicative semilattices.

Let $\mathbf{S} = \langle S; \wedge, \oplus, \to, 0, 1\rangle$ be a $\oplus$-implicative semilattice. Since $\Box x := x \oplus x$ is a nucleus and thus is a modal operator, we can define a binary relation $R_\Box \subseteq X(\mathbf{S}) \times X(\mathbf{S})$ as $(x,y) \in R_\Box$ iff $\Box^{-1}(x) \subseteq y$.

Proposition 5.39. *Let $\mathbf{S}$ be a $\oplus$-implicative semilattice. Then $R_\Box$ is dense, $R_\Box$ is included in the set-theorical inclusion $\subseteq$, and $\sigma(a \oplus b) = \Box_{R_\Box}(\sigma(a) \cup \sigma(b))$, for any $a, b \in S$.*

Proof. As the modal operator $\Box$ is a nucleus we have that $R_\Box \subseteq R_\Box \circ R_\Box$ and $R_\Box$ is included in $\subseteq$. Let $x \in \sigma(a \oplus b)$, i.e., $a \oplus b \in x$. Suppose that $(x,y) \in R_\Box$ but $a, b \notin y$. As y is irreducible and $\mathbf{S}$ is a distributive semilattice, there exists $c \notin y$ such that $a, b \leq c$. So, $a \oplus b \leq c \oplus c = \Box c \in x$, and so $c \in y$, which is a contradiction. Therefore, $a \in y$ or $b \in y$. Then, $\sigma(a \oplus b) \subseteq \Box_{R_\Box}(\sigma(a) \cup \sigma(b))$.

Let $a, b \in S$ and we suppose that $a \oplus b \notin x$. Consider the order ideal $(a \oplus b]$. Then it is easy to see that $\Box^{-1}(x) \cap (a \oplus b] = \emptyset$. Thus there exists an irreducible filter y such that $\Box^{-1}(x) \subseteq y$ and $a \oplus b \notin y$. As $a, b \leq a \oplus b$, we get $a, b \notin y$. So, $x \notin \Box_{R_\Box}(\sigma(a) \cup \sigma(b))$, i.e, $\Box_{R_\Box}(\sigma(a) \cup \sigma(b)) \subseteq \sigma(a \oplus b)$. □

Let $\mathbf{S}$ be a $\oplus$-implicative semilattice. Recall that $\mathrm{Fi}^*(\mathbf{S}) = \left\{F \in \mathrm{Fi}(\mathbf{S}) : \Box^{-1}(F) = F\right\}$ and $X^*(\mathbf{S}) = \{x \in X(\mathbf{S}) : (x, x) \in R_\Box\}$.

Lemma 5.40. *Let* $\mathbf{S}$ *be a* $\oplus$*-implicative semilattice. The following conditions are equivalent:*

1. $x \in X^*(\mathbf{S})$.
2. $x \in \mathrm{Fi}^*(\mathbf{S})$*, and for all* $a, b \in S$*, one has* $a \oplus b \in x$ *iff* $a \in x$ *or* $b \in x$.

Proof. (1) ⇒ (2). Let $a, b \in S$ be such that $a \oplus b \in x$. By Proposition 5.39 we have $x \in \sigma(a \oplus b) = \Box_{R_\Box}(\sigma(a) \cup \sigma(b))$, and as $x \in R_\Box(x)$, we get $x \in \sigma(a) \cup \sigma(b)$, i.e, $a \in x$ or $b \in x$. The converse is immediate because $a, b \leq a \oplus b$. We note that this condition implies that x is irreducible.

(2) ⇒ (1). Let $\Box a \in x$. Since $\Box a = a \oplus 0 \in x$, and $0 \notin x$, we have $a \in x$. So, $\Box^{-1}(x) \subseteq x$, and hence $x \in X^*(\mathbf{S})$. □

Lemma 5.41. *Let* $\mathbf{S}$ *be a* $\oplus$*-implicative semilattice. Let* $F \in \mathrm{Fi}^*(\mathbf{S})$*. If* $a \notin F$*, then there exists* $x \in X^*(\mathbf{S})$ *such that* $F \subseteq x$ *and* $a \notin x$.

Proof. Let $F \in \mathrm{Fi}^*(\mathbf{S})$ and suppose that $a \notin F$. Consider the family

$$\mathcal{F} = \{H \in \mathrm{Fi}^*(\mathbf{S}) : F \subseteq H \text{ and } a \notin H\}.$$

Then $\mathcal{F} \neq \emptyset$. It is clear that we can apply Zorn's Lemma. So, there exists a maximal element x in $\mathcal{F}$. It is easy to see that x is an irreducible filter. Thus, $x \in X^*(\mathbf{S})$. □

Motivated by the duality for implicative semilattices and Proposition 5.39, we introduce the following definition.

Definition 5.42. A $\oplus$*-space* is a structure $\langle X, R, \tau\rangle$ such that

1. $\langle X, \tau\rangle$ is a compact IS-space.
2. $R \subseteq X \times X$ is a dense relation such that $R \subseteq \leq$, where $\leq$ is the specialization order.
3. $R(x)$ is a closed subset of $\langle X, \tau\rangle$, for each $x \in S$.
4. $\Box_R(U \cup V) \in \mathrm{S}(X)$, for any $U, V \in \mathrm{S}(X)$.

Proposition 5.43. *If* $\mathbf{S} = \langle S; \wedge, \oplus, \to, 0, 1\rangle$ *is a* $\oplus$*-implicative semilattice, then* $\langle X(\mathbf{S}), R_\Box, \tau_{\mathbf{S}}\rangle$ *is* $\oplus$*-space, where the topology* $\tau_{\mathbf{S}}$ *is generated by the base* $\{\sigma(a)^c : a \in S\}$.

Proof. That $\langle X(\mathbf{S}), \tau_{\mathbf{S}}\rangle$ is a compact IS-space follows from the duality for bounded implicative semilattices. Let $x \in X$. Since $R(x) = \bigcap\{\sigma(a) : \Box a \in x\}$, we get that $R(x)$ is a closed subset of $\langle X(\mathbf{S}), \tau_{\mathbf{S}}\rangle$. By the properties of $\Box$, we have that $R_\Box$ is dense and $R_\Box \subseteq\leq$.

We prove (4). Let $U, V \in \mathrm{S}(X)$. Then there exist $a, b \in S$ such that $U = \sigma(a)$ and $V = \sigma(b)$. So, by Proposition 5.39

$$\Box_{R_\Box}(U \cup V) = \Box_{R_\Box}(\sigma(a) \cup \sigma(b)) = \sigma(a \oplus b).$$

Thus, $\Box_{R_\Box}(U \cup V) \in \mathrm{S}(X)$.

Finally the condition (4) follows by Proposition 5.39. □

Proposition 5.44. *Let $\langle X, R, \tau\rangle$ be a $\oplus$-space. Then $\langle \mathrm{S}(X); \cap, \oplus, \to, \emptyset, \rangle$ is a $\oplus$-implicative semilattice, where $\oplus$ is given by $U \oplus V = \Box_R(U \cup V)$, for each $U, V \in \mathrm{S}(X)$, and the map $H_X : X \to X(\mathbf{S}(X))$ is a homeomorphism between the $\oplus$-spaces $\langle X, R, \tau\rangle$ and $\langle X(\mathrm{S}(X)), R_{\Box_R}, \tau_{\mathrm{S}(X)}\rangle$ satisfying the condition*

$$(x, y) \in R \text{ iff } (H_X(x), H_X(y)) \in R_{\Box_R},$$

for every $x, y \in X$.

Proof. Since $U \oplus \emptyset = \Box_R(U)$, and R is serial and dense, $R \subseteq \leq$, we get that $\langle \mathrm{S}(X), \cap, \Box_R, \to, \emptyset, X\rangle$ is a bounded implicative semilattice with a nucleus $\Box_R$. It is immediate to see that the other conditions of Definition 3.14 are satisfied. By the duality for compact IS-spaces we have that the map $H_X : X \to X(\mathbf{S}(X))$ is a homeomorphism. In the same way as what was done in Proposition 5.32 we can prove that H_X satisfies the condition $(x, y) \in R$ iff $(H_X(x), H_X(y)) \in R_{\Box_R}$, for all $x, y \in X$. □

Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two $\oplus$-implicative semilattices. A $\oplus$-homomorphism is a bounded implicative homomorphism $h : \mathbf{S}_1 \to \mathbf{S}_2$ such that $h(a \oplus b) = h(a) \oplus h(b)$, for every $a, b \in S_1$.

We recall that if a map $h : \mathbf{S}_1 \to \mathbf{S}_2$ between two implicative semilattices $\mathbf{S}_1$ and $\mathbf{S_2}$ is a bounded implicative homomorphism, then $h^{-1}(F)$ is a filter of $\mathbf{S}_1$ for every filter F of $\mathbf{S}_2$. This fact will be used in the following result.

Proposition 5.45. *Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two $\oplus$-implicative semilattices. Let $h : \mathbf{S}_1 \to \mathbf{S}_2$ be a bounded implicative homomorphism. Then the following conditions are equivalent:*

1. *$h(a \oplus b) \leq h(a) \oplus h(b)$, for every $a, b \in S_1$.*
2. *$h^{-1}(x) \in X^*(\mathbf{S}_1)$, for every $x \in X^*(\mathbf{S}_2)$.*
3. *For each $x \in X^*(\mathbf{S}_2)$ there exists $y \in X^*(\mathbf{S}_1)$ such that $r_h(x) = [y)$.*

Proof. (1) $\Rightarrow$ (2) Let $x \in X^*(\mathbf{S}_2)$. As h is a bounded implicative homomorphism, then $h^{-1}(x)$ is a proper filter. We will use Lemma 5.40. Let $a, b \in S_1$ such that $a \oplus b \in h^{-1}(x)$. Then $h(a \oplus b) \leq h(a) \oplus h(b) \in x$, and as $(x, x) \in R_\Box$, $h(a) \in x$ or $h(b) \in x$. Then, $a \in h^{-1}(x)$ or $b \in h^{-1}(x)$. We prove that $h^{-1}(x) \in \mathrm{Fi}^*(\mathbf{S}_1)$. Let $a \in \Box^{-1}(h^{-1}(x))$.Then $h(\Box a) = h(a \oplus 0) \leq h(a) \oplus h(0) = h(a) \oplus 0 = \Box h(a) \in x$. As $x \in X^*(\mathbf{S}_2)$, we have $h(a) \in x$. So, $h^{-1}(x) \in \mathrm{Fi}^*(\mathbf{S}_1)$. Thus, $h^{-1}(x) \in X^*(\mathbf{S}_1)$.

(2) $\Rightarrow$ (1) Suppose that there exist $a, b \in S_1$ such that $h(a \oplus b) \nleq h(a) \oplus h(b)$. Then there is $z \in X(\mathbf{S}_2)$ such that $h(a \oplus b) \in z$ and $h(a) \oplus h(b) \notin z$. As $h(a) \oplus h(b) = \Box(h(a) \oplus h(b))$, then $h(a) \oplus h(b) \notin \Box^{-1}(z)$. As $\Box^{-1}(z) \in \mathrm{Fi}^*(\mathbf{S}_2)$, by Lemma 5.41, there exists $x \in X^*(\mathbf{S}_2)$ such that $\Box^{-1}(z) \subseteq x$ and $h(a) \oplus h(b) \notin x$. Then $h(a)$, $h(b) \notin x$. As $\Box^{-1}(z) \subseteq x$ implies $z \subseteq x$, then $h^{-1}(z) \subseteq h^{-1}(x)$. Since $h(a \oplus b) \in z$, we have $a \oplus b \in h^{-1}(x)$. By hypothesis, $h^{-1}(x) \in X^*(\mathbf{S}_1)$. So, $a \in h^{-1}(x)$ or $b \in h^{-1}(x)$, i.e, $h(a) \in x$ or $h(b) \in x$, which is a contradiction. Thus, we have proved that $h(a \oplus b) \leq h(a) \oplus h(b)$, for every $a, b \in S_1$.

The equivalence (2) $\Leftrightarrow$ (3) follows by Remark 5.36 . □

Proposition 5.46. *Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two $\oplus$-implicative semilattices. Let $h : \mathbf{S}_1 \to \mathbf{S}_2$ be a bounded implicative homomorphism. Then the following conditions are equivalent:*

1. *$h(a) \oplus h(b) \leq h(a \oplus b)$, for every $a, b \in S_1$.*
2. *$r_h \circ R_{\Box_1} \subseteq R_{\Box_2} \circ r_h$.*

Proof. (1) $\Rightarrow$ (2) Let $x \in X(\mathbf{S}_2)$, and $y, z \in X(\mathbf{S}_1)$. Suppose that $h^{-1}(x) \subseteq y$ and $(y, z) \in R_{\Box_1}$. We prove that there exists $w \in X(\mathbf{S}_2)$ such that $(x, w) \in R_{\Box_2}$ and $(w, z) \in r_h$.

Consider the order-ideal $(h(z^c)] = \{b \in S_2 : \exists c \notin z\ (b \leq h(c))\}$. We prove that $\Box_2^{-1}(x) \cap (h(z^c)] = \emptyset$. Otherwise, there exists $\Box_2 a \in x$, and $c \notin z$ such that $a \leq h(c)$. Then $\Box_2 a = a \oplus 0 \leq \Box_2 h(c) = h(c) \oplus 0 \leq h(c \oplus 0) = h(\Box_1 c) \in x$. Since $h^{-1}(x) \subseteq y$, $\Box_1 c \in y$, and as$(y, z) \in R_{\Box_1}$, we get $c \in z$, which is an absurd. Thus, there exists $w \in X(\mathbf{S}_2)$ such that $(x, w) \in R_{\Box_2}$ and $h^{-1}(w) \subseteq z$.

(2) $\Rightarrow$ (1) Suppose that there exist $a, b \in S_1$ such that $h(a) \oplus h(b) \nleq h(a \oplus b)$. Then there exists $x \in X(\mathbf{S}_2)$ such that $h(a) \oplus h(b) \in x$ and $h(a \oplus b) \notin x$. So, $a \oplus b \notin h^{-1}(x)$, and as $h^{-1}(x)$ is a filter of $\mathbf{S}_1$ there is $y \in X(\mathbf{S}_1)$, such that $h^{-1}(x) \subseteq y$ and $a \oplus b \notin y$. By Proposition 5.39, there exists $z \in X(\mathbf{S}_1)$ such that $\Box_1^{-1}(y) \subseteq z$, and $a, b \notin z$. By hypothesis, there exists $w \in X(\mathbf{S}_2)$ such that $\Box_2^{-1}(x) \subseteq w$ and $h^{-1}(w) \subseteq z$. Since $h(a) \oplus h(b) = \Box_2 (h(a) \oplus h(b)) \in x$ we get $h(a) \in w$ or $h(b) \in w$. Since $h^{-1}(w) \subseteq z$, we have $a \in z$ or $b \in z$, which is a contradiction. Thus, $h(a) \oplus h(b) \leq h(a \oplus b)$, for every $a, b \in S_1$. $\Box$

Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two $\oplus$-implicative semilattices. Let $h : \mathbf{S}_1 \to \mathbf{S}_2$ be a bounded implicative homomorphism. We shall say that h is a $\oplus$-homomorphism if $h(a \oplus b) = h(a) \oplus h(b)$ for all $a, b \in A$. Let $\mathsf{IS}\oplus$ be the category whose objects are $\oplus$-implicative semilattices and whose morphisms are $\oplus$-homomorphisms.

Let $h : \mathbf{S}_1 \to \mathbf{S}_2$ be a $\oplus$-homomorphism. Since h is a bounded implicative semilattice, then the relation $r_h \subseteq X(\mathbf{S}_2) \times X(\mathbf{S}_1)$ given by $(x, y) \in r_h$ iff $h^{-1}(x) \subseteq y$, for all $(x, y) \in X(\mathbf{S}_2) \times X(\mathbf{S}_1)$ is a functional meet-relation [10].

Definition 5.47. Let $\langle X_i, R_i, \tau_i \rangle$, for $i = 1, 2$, be $\oplus$-spaces. Let $X_i^* = \{x \in X_i : (x, x) \in R_i\}$. Let $r \subseteq X_1 \times X_2$ be a relation. We shall say that r is a $\oplus$-relation if r is a functional meet-relation satisfying the following conditions:

(OR1) $R_1 \circ r \subseteq r \circ R_2$.

(OR2) For each $x \in X_1^*$ there exists $y \in X_2^*$ such that $r(x) = [y)$.

Propositions 5.45 and 5.46 give us the following corollary.

Corollary 5.48. *Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two $\oplus$-semilattices. Let $h : S_1 \to S_2$ be a bounded implicative homomorphism. Consider the functional meet-relation $r_h \subseteq X(\mathbf{S}_2) \times X(\mathbf{S}_1)$ defined by $(x, y) \in r_h$ iff $h^{-1}(x) \subseteq y$. Then h is a $\oplus$-homomorphism iff r_h is a $\oplus$-relation.*

It is easy to see that the composition $*$ between $\oplus$-relations given in Definition 5.3 is a $\oplus$-relation. Thus we have a category ISp$\oplus$ whose objects are $\oplus$-spaces and whose morphism are $\oplus$-relations.

From the categorical duality already established for implicative semilattices in [13, 7, 10], together with Propositions 5.39, 5.43, 5.44, and Corollary 5.48 we have the following result.

Theorem 5.49. *The categories $\mathsf{IS}\oplus$ and ISp$\oplus$ are dually equivalent.*

6. Concluding remarks

The present study has been proposed as a contribution towards a topological understanding of a few classes of intuitionistic modal algebras. These structures, as we have seen, arise as subreducts of nuclear Heyting algebras expanded with certain term definable operations whose definitions can be motivated within the study of fragments of quasi-Nelson logic.

Considering nuclear Heyting algebras in the expanded language $\{\wedge, \vee, \to, \odot, \oplus, \rightharpoonup, \Box, 0, 1\}$, where the operations are interpreted as indicated in the previous sections, we recall that we have here focused on the following fragments:

(i) $\{\odot, \to, 0, 1\}$, corresponding to the nH-semigroups of Subsection 5.2;

(ii) $\{\wedge, \rightharpoonup, 0, 1\}$, corresponding to the weak implicative semilattices of Subsection 5.4;

(iii) $\{\wedge, \oplus, \to, 0, 1\}$, corresponding to the $\oplus$-implicative semilattices of Subsection 5.5.

As mentioned in the Introduction, the above classes of algebras arise as factors in the twist representation of the subreducts of quasi-Nelson algebras corresponding, respectively, to the following fragments:

(i) $\{\Rightarrow^2, \sim\}$, i.e. quasi-Nelson implication algebras [25];

(ii) $\{*, \sim\}$, i.e. quasi-Nelson monoids [27];

(iii) $\{\wedge, \Rightarrow^2, \sim\}$, i.e. quasi-Nelson semihoops [27].

The study developed in the previous sections can be straightforwardly extended to the algebras arising from other fragments of the quasi-Nelson language, such as:

(iv) $\{\wedge, \vee, \neg, \sim\}$, corresponding to the quasi-Kleene algebras with weak pseudo-complement introduced in [26], or twist-algebras over pseudo-complemented lattices endowed with a nucleus;

(v) $\{\wedge, \vee, \sim, 0, 1\}$, corresponding to the quasi-Kleene algebras introduced in [28], or twist-algebras over distributive lattices endowed with a nucleus;

(vi) $\{*, \Rightarrow^2, \sim\}$, corresponding to the quasi-Nelson pocrims introduced in [27], or twist-algebras over nuclear implicative semilattices.

We leave the above as suggestions for future research, and we take this opportunity to indicate two further fragments of the quasi-Nelson language (which have not been singled out yet) as potentially interesting ones:

(vii) $\{\vee, \Rightarrow^2, \sim\}$, whose corresponding algebras (we anticipate) ought to arise as twist-algebras over nH-semigroups expanded with a join operation;

(viii) $\{*, \vee, \sim\}$, whose corresponding algebras (we anticipate) ought to arise as twist-algebras over weak implicative semilattices expanded with a join, i.e. (basic) weak Heyting algebras.

We also believe that a deeper universal algebraic study of the varieties considered in the present paper – in particular a classification of their subvarieties – would also deserve further study.

Lastly, we should like to mention that an investigation of the logics associated to intuitionistic modal algebras might also turn out to be worth pursuing. For most of the classes of algebras at hand, the corresponding assertional logic will be algebraizable in the sense of Blok and Pigozzi, and may therefore be easily studied by algebraic means; in other cases, however – consider e.g. the logic of weak implicative semilattices, but also the order-preserving consequence relation associated to any of the above-mentioned varieties – it is not hard to see that the corresponding logic will not be algebraizable. In any case, the topological considerations developed in the present paper may provide a suitable setting for a study of these logics from the point of view of relational semantics.

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