A UNIQUENESS RESULT FOR A SINGULAR ELLIPTIC EQUATION WITH GRADIENT TERM

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Abstract. In this paper we prove uniqueness of solution for a problem whose simplest model is

$$\begin{cases} -\Delta u + \frac{k}{u}|Du|^2 = f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{0.1}$$

with $k \geq 1$, $0 \lneq f \in L^\infty(\Omega)$ and Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$. So far, uniqueness results were known if $k < 1$ while existence holds for any $k \geq 1$ and f positive in open sets compactly embedded in a neighborhood of the boundary.
We extend the uniqueness results to the case of $k \geq 1$ and we also show, with an example, that existence cannot be true if f is zero near the boundary. We even deal with the uniqueness result when f is replaced by a nonlinear term λu^q with $0 < q < 1$ and $\lambda > 0$.

1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and $0 \lneq f \in L^q(\Omega)$, with $q > \frac{N}{2}$, $N \geq 2$. We consider the boundary value problem

$$\begin{cases} -\Delta u + g(u)|Du|^2 = f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where the function $g : (0, +\infty) \to [0, +\infty)$ is unbounded at zero. Actually it is a classical issue the study of this kind of singular elliptic equations with gradient terms. For the existence of a solution $0 < u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ two thresholds appear naturally: the first one relies to be an integrability condition on $e^{-\int_1^s g(t)dt}$ as $s \sim 0$, i.e. whether or not it is satisfied that

$$e^{-\int_1^s g(t)dt} \in L^1(0,1). \tag{1.2}$$

Under this first threshold, a solution $u \in H_0^1(\Omega)$ for (1.1) exists even in a more general context (say, with a nonlinear differential operator instead of the Laplacian) and with a non negative non trivial datum $f \in L^{\frac{2N}{N+2}}(\Omega)$ on the right hand side (see [8], for instance).

In order to prove existence of a solution relaxing the growth condition on g at 0 and overcoming the first threshold, i.e. when condition (1.2) is not satisfied, the datum $f(x)$ cannot degenerate to 0 inside Ω. More precisely f has to satisfy

$$f(x) \geq c_\omega > 0, \qquad \forall \omega \subset\subset \Omega\,. \tag{1.3}$$

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Then a second natural threshold turns up which is

$$\int_0^1 \sqrt{g(s)}ds < +\infty\,. \tag{1.4}$$

In [1] it is proved that the existence of a solution holds if (1.3) and (1.4) are satisfied. Otherwise (see [1] and [16]) nonexistence of $H_0^1(\Omega)$ solutions occurs. In fact, as was showed in [11], condition (1.3) is only needed in a neighborhood of the boundary, i.e. it can be replaced by

$$f(x) \geq c_\omega > 0 \quad \text{for any } \omega \subset \Omega_\delta\,, \tag{1.5}$$

where $\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \delta\}$, for some $\delta > 0$.

In [13] lower order terms of the form $g(u)|Du|\chi_{\{u>0\}}$ were considered and solutions that can be zero on a closed "large" set were obtained, even if g is negative. Solutions that change sign in the interior of Ω have also been obtained in [14] if g is negative and f changes sign.

As far as uniqueness of solutions for (1.1) is concerned, the only known results about singular gradient terms rely to be [3] and [6] that cover the case in which (1.2) is satisfied. Up to our knowledge, it was unknown if it is possible to go beyond the first threshold by proving uniqueness of solution.

The purpose of this paper is to give a way of comparing a subsolution and a supersolution for (1.1) when we are beyond the threshold (1.2). The main difficulty for such a problem is represented by a lack of Hopf Lemma due to the presence of the singular gradient term. Here it is proved that solutions (via a suitable comparison with sub and supersolutons) behave at the boundary as a suitable power of the distance to the boundary. This forces us to give a more accurate estimate of solutions near $\partial\Omega$ that gets rid of the singularity of the lower order term.

The paper is organized as follows. Section 2 is devoted to prove a uniqueness result for smooth solutions of (1.1) in the case where condition (1.2) is not satisfied. We also assume (1.5) which, as mentioned above, is needed to prove existence results if (1.2) fails. Actually, in Section 3, we prove with a counterexample that this condition is in some sense necessary for the existence.

2. Comparison principle and uniqueness

In this section we prove a comparison principle for problem (1.1) in the case in which $g : (0, +\infty) \to [0, +\infty)$ is a continuous function, singular at 0, such that $g(s) \simeq k/s$, $k \geq 1$. More precisely we assume the function g satisfies the following hypothesis:

$$\exists a_0 > 0 \quad \text{and } \kappa \geq 1 : \qquad \max\{\kappa - 1, 1\} \leq g(s)s \leq \kappa,\ s \in (0, a_0). \tag{2.1}$$

Observe that, without loss of generality, we may assume that $\kappa > 1$. Moreover, the above hypothesis implies that

$$e^{-G(s)} \notin L^1(0, a_0), \quad \text{where} \qquad G(s) = \int_{a_0}^s g(t)dt.$$

As far as the right hand side of the equation in (1.1) is concerned, we assume that $f : \Omega \to [0, +\infty)$ verifies $f \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\exists \delta > 0 \quad \exists \ \ 0 < \alpha \leq \beta,\ r \geq 0 : \qquad \alpha\varphi_1^r(x) \leq f(x) \leq \beta\varphi_1^r(x)\,,\ \text{in } \Omega_\delta, \tag{2.2}$$

where φ_1 is a positive eigenfunction corresponding to the first eigenvalue associated to $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions. We point out that (2.2) implies that, at least in the neighborhood of the boundary

$$\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\},$$

the function f satisfies (1.5).

In the following, given Ω a bounded open set of $\mathbb{R}^N$, with $N \geq 2$, we denote by $H^1_c(\Omega)$ the space of functions that belong to $H^1_0(\Omega)$ with compact support in Ω. First we recall the meaning that we give to a subsolution and a supersolution of the singular equation

$$-\Delta u + g(u)|Du|^2 = f(x) \qquad \text{in } \Omega. \tag{2.3}$$

Definition 2.1. Assume that $f \in L^1_{\text{loc}}(\Omega)$, then:

- we say that $u \in H^1(\Omega) \cap C^0(\overline{\Omega})$ is a *subsolution* to (2.3) if $u > 0$

$$\int_\Omega Du \cdot D\phi + \int_\Omega g(u)|Du|^2\phi \leq \int_\Omega f(x)\phi \qquad \forall \phi \in H^1_c(\Omega) \cap L^\infty(\Omega)\,, \quad \phi \geq 0\,. \tag{2.4}$$

- we say that $v \in H^1(\Omega) \cap C^0(\overline{\Omega})$ is a *supersolution* to (2.3) if $v > 0$ and

$$\int_\Omega Dv \cdot D\phi + \int_\Omega g(v)|Dv|^2\phi \geq \int_\Omega f(x)\phi \qquad \forall \phi \in H^1_c(\Omega) \cap L^\infty(\Omega)\,, \quad \phi \geq 0\,. \tag{2.5}$$

- We say that $z \in H^1_0(\Omega) \cap C^0(\overline{\Omega})$ is a *solution* for (1.1) if it is both a subsolution and a supersolution for (2.3).

Remark 2.2. Let us stress that the definition of solution (as well as the ones of sub and supersolution) requires that it belongs to $C^0(\overline{\Omega})$, that seems to be a restrictive assumption. Anyway, as well explained in [1] (Remark 2.6), being solutions of (1.1) subsolutions of the Dirichlet problem associated to $-\Delta u = f$ in Ω, following [15], we deduce that if $f \in L^m(\Omega)$ with $m > \frac{N}{2}$, that u is also continuous in $\overline{\Omega}$.

The main result we want to prove is the following.

Theorem 2.3. *Under the assumptions* (2.1) *and* (2.2)*, there exists a unique positive* $H^1_0(\Omega) \cap C^0(\overline{\Omega})$ *solution to* (1.1).

The result stated above is the direct consequence of a comparison result between supersolutions and subsolutions to (2.3) with a suitable boundary behavior. In fact, the main effort will be to prove that any sub/supersolution has the same behavior at $\partial\Omega$.

Thus we begin by defining some auxiliary functions and collecting together their main properties. We have already defined the function $G(s) = \int_{a_0}^s g(t)dt$, for some $a_0 > 0$. Let us also consider the function $\psi(s)$ defined by

$$\psi(s) = \int_{a_0}^s e^{-G(t)}dt, \quad s > 0. \tag{2.6}$$

Observe that $\psi(s)$ is an increasing function and, assuming (2.1),

$$\psi_0 \equiv \lim_{s\to 0^+} \psi(s) = -\int_0^{a_0} e^{-G(t)}dt = -\infty\,,$$

and

$$\psi_\infty \equiv \lim_{s\to +\infty} \psi(s) = \int_{a_0}^{+\infty} e^{-G(t)}dt > 0\,,$$

being ψ_∞ possibly infinite.

For our purpose, we define a third function, with two parameters: for any $a, b > 0$ we define $\varphi_{a,b}(s)$ (φ for brevity) as the solution of the following Cauchy problem:

$$\begin{cases} \varphi''(s) = \big[g(\varphi(s))\varphi'(s) - g(s)\big]\varphi'(s) & s \in (0, a_0)\,, \\ \varphi(a_0) = a\,, \\ \varphi'(a_0) = b\,. \end{cases} \tag{2.7}$$

Observe that φ solves

$$\varphi'(s) = b\; e^{G(\varphi(s)) - G(s) - G(a)} \qquad \text{in } (0, a_0)\,.$$

Thus we have that

$$\varphi'(s) e^{-G(\varphi(s))} = b\; e^{-G(s) - G(a)} \qquad \text{in } (0, a_0)\,,$$

and so, integrating the above identity between a_0 and s, we deduce that

$$\psi(\varphi(s)) - \psi(\varphi(a_0)) = \int_{\varphi(a_0)}^{\varphi(s)} e^{-G(t)} dt = b\; e^{-G(a)} \int_{a_0}^{s} e^{-G(t)} dt = b\; e^{-G(a)} \psi(s)\,.$$

i.e.

$$\psi(\varphi(s)) = \psi(a) + b\; e^{-G(a)} \psi(s)\,. \tag{2.8}$$

In the following lemma we list some useful properties of function φ when condition (2.1) is satisfied.

Lemma 2.4. *Under the hypothesis* (2.1)*, the function* φ *defined in* (2.7) *satisfies:*

1. $\varphi(s)$ *is well defined for every* $s \in (0, a_0)$*;*
2. $\lim_{s \to 0^+} \varphi(s) = 0$*;*
3. $\varphi \in C^2(0, a_0)$ *and* $\varphi''(s) = (g(\varphi(s))\varphi'(s) - g(s))\varphi'(s)$ *for every* $s \in (0, a_0)$*;*
4. *If* $a < a_0$ *and* $b\; e^{-G(a)} \geq 1$ *then* $0 < \varphi(s) \leq s$ *for every* $s \in (0, a_0)$ *and* φ' *is positive and bounded;*
5. *If* $b\; e^{-G(a)} \geq (1+\varepsilon)^{\kappa - 1}$ *and* $a \leq \frac{a_0}{1+\varepsilon}$ *for some* $\varepsilon > 0$ *then* $0 < (1+\varepsilon)\varphi(s) \leq s$ *for every* $s \in (0, a_0)$*;*
6. *If* $b\; e^{-G(a)} = (1+\varepsilon)^{\kappa - 1}$ *and* $a = \frac{a_0}{1+\varepsilon}$ *then* $\varphi'(s) \leq 1$ *for every* $s \in (0, a_0)$*.*

Proof. Items $(1) - (3)$ are straightforward. In order to prove item (4) we observe that,

$$\psi(a) + b\; e^{-G(a)} \psi(s) \leq \psi(s),\ s \in (0, a_0).$$

Thus, since ψ^{-1} is increasing we deduce that $\varphi(s) \leq s$. Taking into account that $\varphi'(s) = b\; e^{-G(a)} e^{G(\varphi(s)) - G(s)}$ we deduce also that φ' is positive and bounded.

Now we prove item (5). This is deduced from the inequality

$$\psi(a) + b e^{-G(a)} \psi(s) \leq \psi\left(\frac{1}{1+\varepsilon} s\right),\ s \in (0, a_0).$$

Indeed, the function $h(s) = \psi(a) + b\; e^{-G(a)} \psi(s) - \psi\left(\frac{1}{1+\varepsilon} s\right)$ is increasing since

$$e^{G(s)} h'(s) = b\; e^{-G(a)} - \frac{1}{1+\varepsilon} e^{G(s) - G\left(\frac{s}{1+\varepsilon}\right)}.$$

Observe that if $g(s)s \leq \kappa$ then

$$G(s) - G\left(\frac{s}{1+\varepsilon}\right) = \int_{\frac{s}{1+\varepsilon}}^{s} g(t) dt \leq \ln{(1+\varepsilon)^\kappa}\,.$$

In particular

$$e^{G(s)}h'(s) \geq be^{-G(a)} - (1+\varepsilon)^{\kappa-1} \geq 0.$$

Thus $h(s) \leq h(a_0) = \psi(a) - \psi\left(\frac{a_0}{1+\varepsilon}\right) \leq 0$, $s \in (0, a_0)$.

Finally, item (6) is deduced since $g(s)s \geq \kappa - 1$ implies that

$$\begin{aligned}\varphi'(s) = (1+\varepsilon)^{\kappa-1}e^{G(\varphi(s))-G(s)} &\leq (1+\varepsilon)^{\kappa-1}e^{G\left(\frac{s}{1+\varepsilon}\right)-G(s)} \\ &\leq (1+\varepsilon)^{\kappa-1}e^{-(\kappa-1)\ln(1+\varepsilon)} = 1.\end{aligned}$$

□

Remark 2.5. Let us explicitly compute the functions introduced in (2.6) and (2.7) in the case $g(s) = k/s$, with $k \geq 1$, that is our model case (see (0.1)). We have that

$$\psi(s) = \frac{a_0^k}{k-1}\left(a_0^{1-k} - s^{1-k}\right) \quad \text{and} \quad \varphi(s) = \left(b\left(\frac{a_0}{a}\right)^k\left(s^{1-k} - a_0^{1-k}\right) + a^{1-k}\right)^{-\frac{1}{k-1}}$$

if $k > 1$, while

$$\psi(s) = a_0 \log\left(\frac{s}{a_0}\right) \quad \text{and} \quad \varphi(s) = a\left(\frac{s}{a_0}\right)^{\frac{b}{a}a_0}$$

if $k = 1$.

Now we are ready to prove the following comparison result among subsolutions and supersolutions of (2.3) that have a comparable behavior at $\partial\Omega$. We denote $d(x) = \operatorname{dist}(x, \partial\Omega)$.

Theorem 2.6. *Let u, v be respectively a subsolution and a supersolution of* (2.3) *with g satisfying* (2.1) *and $f \in L^1(\Omega)$, $f \geq 0$. Suppose that*

$$\liminf_{d(x)\to 0} \frac{v(x)}{u(x)} \geq 1\,. \tag{2.9}$$

Then $u \leq v$ in Ω.

Remark 2.7. Although the main interest of the previous theorem, due to the singularity, is when $u = v = 0$ on $\partial\Omega$ we observe that the above result does not depend on the fact that u and v vanish at $\partial\Omega$.

Proof. First, we fix $a_0 > \|u\|_{L^\infty(\Omega)}$ and for any $\varepsilon > 0$ we denote by φ_ε the function $\varphi_{a,b}$ with $a = \frac{a_0}{1+\varepsilon}$ and $b\ e^{-G(a)} = (1+\varepsilon)^{\kappa-1}$. Recall that, under assumption (2.1) we have that $\varphi_\varepsilon(a_0) = \frac{a_0}{1+\varepsilon}$, $\varphi_\varepsilon'(a_0)e^{-G(\varphi_\varepsilon(a_0))} = (1+\varepsilon)^{\kappa-1}$ and

$$\varphi_\varepsilon(s) = \psi^{-1}\left(\psi\left(\frac{a_0}{1+\varepsilon}\right) + (1+\varepsilon)^{\kappa-1}\psi(s)\right). \tag{2.10}$$

We claim that the function $u_\varepsilon = \varphi_\varepsilon(u)$ is still a subsolution of (2.3) which satisfies:

$$\liminf_{d(x)\to 0} \frac{u}{u_\varepsilon} \geq (1+\varepsilon). \tag{2.11}$$

Indeed, using Lemma 2.4, we have that $u_\varepsilon = \varphi_\varepsilon(u) \in H^1(\Omega)\cap C^0(\overline{\Omega})$, $u_\varepsilon > 0$ and it verifies

(2.12)

$$\begin{aligned}-\Delta u_\varepsilon + g(u_\varepsilon)|Du_\varepsilon|^2 - f(x) \leq \\ &\leq -f(x)[1-\varphi_\varepsilon'(u)] + |Du|^2\Big[-g(u)\varphi_\varepsilon'(u) - \varphi_\varepsilon''(u) + g(\varphi_\varepsilon(u)){\varphi'}_\varepsilon^2(u)\Big] \\ &= -f(x)[1-\varphi_\varepsilon'(u)].\end{aligned}$$

Using that $\varphi_\varepsilon'(u)\leq 1$ we deduce that the right hand side in (2.12) is nonpositive and consequently u_ε is a subsolution.

The second part of the statement is deduced from item (5) of Lemma 2.4. Indeed, since $(1+\varepsilon)\varphi_\varepsilon(u)\leq u$ then

$$\liminf_{d(x)\to 0}\frac{u}{\varphi_\varepsilon(u)} \geq (1+\varepsilon), \tag{2.13}$$

and the claim is proved.

Combining (2.9) with (2.11) we deduce that

$$\liminf_{d\to 0}\frac{v}{u_\varepsilon} = \liminf_{d\to 0}\frac{v}{u}\frac{u}{u_\varepsilon} \geq (1+\varepsilon)\liminf_{d\to 0}\frac{v}{u} > 1$$

and consequently, there exists $\Omega_\varepsilon\subset\Omega$ such that $u_\varepsilon - v\leq 0$ in Ω_ε.

Thus, with ψ defined in (2.6), we have that $\big[\psi(u_\varepsilon)-\psi(v)\big]^+$ is supported in $\Omega\setminus\Omega_\varepsilon$. This implies that the function $\phi = e^{-G(u_\varepsilon)}\big[\psi(u_\varepsilon)-\psi(v)\big]^+$ belongs to $H^1_c(\Omega)\cap L^\infty(\Omega)$ and it can be taken as test function in the formulation of u_ε, so that

$$\int_\Omega D\psi(u_\varepsilon)\cdot D\big[\psi(u_\varepsilon)-\psi(v)\big]^+ \leq \int_\Omega f(x)e^{-G(u_\varepsilon)}\big[\psi(u_\varepsilon)-\psi(v)\big]^+. \tag{2.14}$$

Analogously for the supersolution, we take $e^{-G(v)}\big[\psi(u_\varepsilon)-\psi(v)\big]^+$ as test function, and we have that

$$\int_\Omega D\psi(v)\cdot D\big[\psi(u_\varepsilon)-\psi(v)\big]^+ \geq \int_\Omega f(x)e^{-G(v)}\big[\psi(u_\varepsilon)-\psi(v)\big]^+. \tag{2.15}$$

Thus, subtracting the two inequalities, we deduce that

$$\int_\Omega |D\big[\psi(u_\varepsilon)-\psi(v)\big]^+|^2 \leq \int_\Omega f(x)\big[\psi'(u_\varepsilon)-\psi'(v)\big]\big[\psi(u_\varepsilon)-\psi(v)\big]^+ \leq 0\,,$$

and consequently $u_\varepsilon\leq v$ also in $\Omega\setminus\Omega_\varepsilon$, $\forall\varepsilon>0$. Letting ε vanish, we have that $u\leq v$ in Ω. □

Remark 2.8. Let us stress that we are left with the case $k=1$ in (0.1). In this case it is not so hard to prove that $\varphi_\varepsilon(s)$ is replaced by $s^{1+\varepsilon}$, i.e. if u is a subsolution, then $u_\varepsilon = u^{1+\varepsilon}$ is still a subsolution.

Indeed, let assume that $\|u\|_{L^\infty(\Omega)}\leq a<1$ (otherwise we can reduce to this case after a rescaling of the equation), and define $u_\varepsilon = u^{1+\varepsilon}$, for $\varepsilon>0$. Hence

$$-\Delta u_\varepsilon + \frac{1}{u_\varepsilon}|Du_\varepsilon|^2 - f(x) \leq -f(x)[1-(1+\varepsilon)u^\varepsilon] \leq 0\,,$$

i.e. u_ε is a subsolution. Moreover, if $u=0$ on $\partial\Omega$, we observe that

$$\liminf_{d(x)\to 0}\frac{u}{u_\varepsilon} = +\infty. \tag{2.16}$$

The next step in order to prove uniqueness of the solution of (1.1), is to prove that, if f satisfies (2.2), all the solutions have a boundary behavior that can be comparable with φ_1^{2+r}, where, as mentioned in the introduction, φ_1 denotes a positive eigenfunction corresponding to the first eigenvalue associated to $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions. We recall the following result concerning with φ_1.

Lemma 2.9. *Let $\gamma > 0$, then there exists $0 < \gamma_1 \leq \gamma_2 < +\infty$ such that*

$$\gamma_1 \leq \ \varphi_1^2 + \gamma |D\varphi_1|^2 \leq \gamma_2 \qquad \textit{in} \ \ \Omega\,.$$

Proof. The second inequality is trivial, since we can choose $\gamma_2 = \|\varphi_1\|^2_{L^\infty(\Omega)} + \gamma \|D\varphi_1\|^2_{L^\infty(\Omega)}$. For the other one, we argue by contradiction. Assume that $\inf_\Omega \varphi_1^2 + \gamma |D\varphi_1|^2 = 0$. Being $\varphi_1^2 + \gamma|D\varphi_1|^2$ smooth in $\overline{\Omega}$, it means that its minimum is equal to zero. Anyway, the minimum point cannot ~~be~~ lie at the interior of Ω, since there $\varphi_1 > 0$. On the other hand at $\partial\Omega$ we have (Hopf Lemma) that $\frac{\partial \varphi_1}{\partial \nu}\big|_{\partial\Omega} < 0$, and consequently $|D\varphi_1|^2 > 0$ at $\partial\Omega$. □

Let us now state our boundary behavior result.

Theorem 2.10. *Let u be any solution to* (1.1) *with g and f satisfying respectively* (2.1) *and* (2.2). *Then there exist $M, \eta > 0$ depending only on Ω and $r \geq 0$, such that*

$$\eta \leq \frac{u}{\varphi_1^{r+2}} \leq M \qquad \textit{in} \ \ \Omega_\delta\,.$$

Proof. The idea is to exploit the comparison principle in Theorem 2.6.

First, consider the following supersolution in Ω_δ of (2.3): $\overline{v}_\varepsilon = M(\varepsilon + \varphi_1)^{r+2}$, for some $M > 0$, and $\varepsilon > 0$. Using that f satisfies (2.2), direct computations show that it turns out to be a supersolution, once that M is chosen sufficiently large, uniformly with respect to ε. Moreover, since for any $\varepsilon > 0$, $\overline{v}_\varepsilon = M\ \varepsilon^{r+2}$ at $\partial\Omega$, we have for any solution u of (1.1)

$$\liminf_{d(x)\to 0} \frac{\overline{v}_\varepsilon}{u} = +\infty\,, \qquad \liminf_{d(x)\to \delta} \frac{\overline{v}_\varepsilon}{u} \geq M \max_{d(x)=\delta} \frac{(\varphi_1(x))^{r+2}}{u(x)} > 1.$$

Thus, thanks to Theorem 2.6, $\overline{v}_\varepsilon \geq u$ in Ω_δ. Letting ε go to 0, we deduce that $u \leq M\varphi_1^{r+2}$ in Ω_δ.

Now we consider for any $0 < \rho < \delta$, $\Omega_\delta^\rho = \{x \in \Omega \,:\, \rho < d(x) < \delta\}$ and observe that u is supersolution of the problem

$$\text{(2.17)} \qquad \begin{cases} -\Delta z + g(z)|Dz|^2 = f(x) & \text{in } \Omega_\delta^\rho\,, \\ z = 0 & \text{on } d(x) = \rho\,, \\ z = u & \text{on } d(x) = \delta\,, \end{cases}$$

We denote by $\varphi_{1,\rho}$ the first positive eigenfunction, with $\|\varphi_{1,\rho}\|_{L^\infty(\Omega)} = 1$, associated to $-\Delta$ in $\Omega^\rho = \{x \in \Omega \,:\, \rho < d(x)\}$ with zero Dirichlet boundary conditions, extended to 0 in $\Omega \setminus \Omega^\rho$, and $\lambda_{1,\rho}$ its corresponding eigenvalue. Observe that $\varphi_{1,\rho} \to \varphi_1$ in $C^0(\overline{\Omega}) \cap C^1_{\text{loc}}(\Omega)$, due to the linearity of the problem and since both φ_1 and $\varphi_{1,\rho}$ are normalized in L^∞ to 1. Moreover $\lambda_{1,\rho} \to \lambda_1$ as $\rho \to 0$.

We can choose $\eta > 0$ independent on ρ such that the function $\underline{u}_\rho = \eta\varphi_{1,\rho}^{r+2}$ is a subsolution of (2.17) for every $\rho \in [0, \delta_0]$ for some $\delta_0 \in (\rho, \delta)$. Indeed, note that

$$-\Delta\underline{u}_\rho + g(\underline{u}_\rho)|D\underline{u}_\rho|^2 - f(x) \le 2\eta\lambda_{1,\rho}\varphi_{1,\rho}^q\left[\varphi_{1,\rho}^2 + \frac{2\kappa-1}{\lambda_{1,\rho}}|D\varphi_{1,\rho}|^2\right] - \alpha\varphi_{1,\rho}^r \quad \text{in } \Omega_\delta^\rho,$$

so that $\underline{u}_\rho$ is a subsolution of (2.17) if

$$0 < \eta \le \min\left\{\frac{\alpha}{2\ \lambda_{1,\rho}\ c_1}, \inf_{d(x)=\delta} u(x)\right\}$$

where

$$c_1 = \max_{\rho\in[0,\delta]}\max_{x\in\overline{\Omega^\rho}}\left[\varphi_{1,\rho}^2 + \frac{2\kappa-1}{\lambda_{1,\rho}}|D\varphi_{1,\rho}|^2\right],$$

(see Lemma above).
Observe that $0 < c_1 < \infty$ since $\varphi_{1,\rho} \to \varphi_1$ in $C^0(\overline{\Omega}) \cap C^1_{\text{loc}}(\Omega)$.

Applying now Theorem 2.6 we deduce that $\underline{u}_\rho \le u$ in Ω_δ^ρ, being in this case $\liminf\limits_{d_\rho\to 0}\dfrac{u}{\underline{u}_\rho} = +\infty$.

Thus letting ρ vanish, we get that

$$\eta\varphi_1^{r+2} = \underline{u}_0 = \lim_{\rho\to 0}\eta\varphi_{1,\rho}^{r+2} \le u \quad \text{in } \Omega_\delta.$$

□

Now we are ready to prove a uniqueness result for solutions of (1.1).

Proof of Theorem 2.3. Assume that problem (1.1) has two solutions u, v: according to Theorem 2.10,

$$0 < \frac{\eta}{M} \le \frac{u}{v} \le \frac{M}{\eta} \qquad \text{in } \Omega_\delta\,.$$

Fix any $\varepsilon > 0$, and consider the function φ_ε defined as

$$\varphi_\varepsilon(s) = \psi^{-1}\left(\psi\left(\frac{a_0}{1+\varepsilon}\right) + (1+\varepsilon)^{\kappa-1}\psi(s)\right),$$

for some $a_0 > \|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}$ and with ψ as in (2.6). Define also, for any $m \in \mathbb{N}$,

$$u_\varepsilon = \varphi_\varepsilon^{[m]}(u) \qquad \text{where } \varphi_\varepsilon^{[m]}(s) = \underbrace{\varphi_\varepsilon \circ \varphi_\varepsilon \circ \cdots \circ \varphi_\varepsilon}_{m\text{-times}}(u)\,.$$

First observe that, given any subsolution $u \in H_0^1(\Omega) \cap C^0(\overline{\Omega})$ of (1.1), by induction any $\varphi_\varepsilon^{[m]}(u)$, with $m \in \mathbb{N}$ is still an $H_0^1(\Omega) \cap C^0(\overline{\Omega})$ subsolution. Moreover $\varphi_\varepsilon^{[m]}(u)$ also satisfies

$$\liminf_{d(x)\to 0^+}\frac{v}{u_\varepsilon} \ge (1+\varepsilon)^m \ \liminf_{d(x)\to 0^+}\frac{v}{u} \ge \frac{\eta}{M}(1+\varepsilon)^m\,;$$

notice that m can be chosen (depending only on η and M) such that the right hand side above is strictly bigger than 1. We can now apply Theorem 2.6 and we have that $u_\varepsilon \le v$; letting ε vanish, we deduce that $u \le v$. Changing the roles of u and v, we get the reverse inequality. □

In particular we have the following Corollary.

Corollary 2.11. *There exists a unique positive $H^1(\Omega)\cap C^0(\overline{\Omega})$ solution to*

$$\begin{cases} -\Delta u + \frac{k}{u}|Du|^2 = f & \text{in } \Omega\,,\\ u = 0 & \text{on } \partial\Omega\,,\end{cases}$$

with $k \geq 1$ and f as in (2.2). □

With the same technique we can prove uniqueness if we replace the right hand side by a power nonlinearity. More specifically, we want to prove the following result.

Theorem 2.12. *Assume that $u, v \in H_0^1(\Omega)\cap C^0(\overline{\Omega})$ are two solutions of*

$$\begin{cases} -\Delta z + g(z)|Dz|^2 = a(x)z^q & \text{in } \Omega\,,\\ z = 0 & \text{on } \partial\Omega\,,\end{cases} \tag{2.18}$$

with $q \in (0,1)$ and $0 < a_1 \leq a(x) \leq a_2$ and g as in (2.1). Then $u \equiv v$ in Ω.

Proof. We reproduce the same proof of Theorems 2.3 and 2.10.

We first want to argue as in Theorem 2.6 in order to deduce that any pair of sub and supersolutions, in the sense of Definition 2.1 with $f(x)$ replaced by $a(x)u^q$ in (2.4) and by $a(x)v^q$ in (2.5), being comparable at $\partial\Omega$ (i.e. such that (2.9) holds true) are ordered in Ω. Hence it is not hard to see that that, given any subsolution u, using items (4)–(6) of Lemma 2.4, we have that $\varphi_\varepsilon(u)$ is still a subsolution, for every $\varepsilon > 0$, with (2.13) in force. Hence we deduce the comparison near $\partial\Omega$.

Moreover, thanks once again to (6) of Lemma 2.4, we can argue as in (2.14)–(2.15) and we also deduce that the comparison still holds at the interior.

Thus, given any sub and supersolutions, we want to prove that necessarily (2.9) is in force. To do this, we just find a suitable couple of sub and supersolutions that behaves in a comparable way at $\partial\Omega$. Observe that, using the regularity properties of φ_1, one can prove that

$$\exists\, 0 < \alpha \leq \beta \;:\quad 0 < \alpha \leq \frac{2\lambda_1}{1-q}\varphi_1^2 + \frac{2}{1-q}|D\varphi_1|^2 \leq \beta\,. \tag{2.19}$$

We set $\overline{u} = M(\varepsilon + \varphi_1)^{\frac{2}{1-q}}$, with $M > 0$ (to be chosen) and $\varepsilon > 0$, so that

$$-\Delta\overline{u} + \frac{1}{\overline{u}}|D\overline{u}|^2 - a(x)\overline{u}^q \geq M\varphi_1^{\frac{2q}{1-q}}\left[\frac{2\lambda_1}{1-q}\varphi_1^2 + \frac{2}{1-q}|D\varphi_1|^2 - a_2\; M^{q-1}\right] \quad \text{in } \Omega.$$

Since $q - 1 < 0$, it is sufficient to chose $M \geq \left(\frac{\alpha}{a_2}\right)^{\frac{1}{q-1}}$ and $\overline{u}$ turns out to be a supersolution to (2.18). Moreover $\overline{u} = M\varepsilon^{\frac{2}{1-q}} > 0$ at $\partial\Omega$.

Arguing as in Theorem 2.10 we can also construct a subsolution in a neighborhood of the boundary, that behaves as $\underline{u} = \eta\varphi_1^{\frac{2}{1-q}}$, with $\eta > 0$, with $\eta \leq \left(\frac{\alpha}{a_1}\right)^{\frac{1}{q-1}}$.

We can now repeat the same proof of Theorem 2.3, and we get uniqueness. □

3. A counterexample

We prove here that condition (1.5) cannot be relaxed, at least in dimension 1, if $e^{-G(s)}$ is not integrable at zero, in order to obtain existence of solution. This is consequence of the following result.

Theorem 3.1. *Assume that $f \in C(0,1)$, $f(x) \geq 0$, $f \not\equiv 0$ and $f(x) \equiv 0$ if $x \in (0,\varepsilon)$ for some $\varepsilon \in (0,1)$. There is no positive solution $u \in H_0^1(0,1)$ of the problem*

$$-u''(x) + \frac{k}{u(x)}\, u'^2(x) = f(x), \qquad x \in (0,1)\,, \qquad u(0) = u(1) = 0.$$

Proof. We argue by contradiction assuming that such a solution exists. Then

$$-u''(x) + \frac{k}{u(x)} u'(x)^2 = 0, \quad x \in (0,\varepsilon) \quad \text{for some } \varepsilon > 0\ .$$

We first recall that a solution has to be positive in $(0,1)$, and moreover that $H^1(0,1)$ is embedded in $AC(0,1)$. Consequently, we deduce that $u(x) \in W^{2,1}(0,1)$, that implies that $u' \in AC$ and so by using once again the equation, that $u \in C^2(0,1)$.

In particular u is a convex function in $(0,\varepsilon)$ and then u' is nondecreasing in this interval which assures the existence of $u'(0)$. Moreover, $u'(0) \geq 0$, since otherwise u would be negative in a (right) neighborhood of 0 (since $u(0) = 0$). If $u'(t) = 0$ for some $t \in (0,\varepsilon)$ then $u'(x) \equiv 0$ for every $x \in (0,t)$ an this implies that $u(x) = u(0) = 0$ for every $x \in (0,t)$, which contradicts that $u > 0$. Thus we may assume that $u'(x) > 0$ for every $x \in [0,\varepsilon)$.

Multiplying the equation by $1/u'(x)$ we have

$$k\,(\ln u(x))' = k\frac{u'(x)}{u(x)} = \frac{u''(x)}{u'(x)} = (\ln u'(x))', \quad x \in (0,\varepsilon),$$

or equivalently, for some positive constant $c > 0$, $\frac{u'(x)}{u^k(x)} = c$ for every $x \in (0,\varepsilon)$.

If $k = 1$ this implies that there exists a positive constant $A > 0$ such that $u(x) = Ae^{cx}$ for every $x \in (0,\varepsilon)$. Thus the claim follows since the condition $u(0) = 0$ is violated.

If, otherwise, $k > 1$ we integrate the differential identity $\frac{u'(x)}{u^k(x)} = c$ and we deduce that

$$\frac{1}{1-k} u^{1-k}(x) = cx + c_1 \quad c > 0\,, c_1 \in \mathbb{R}\,,$$

that is not compatible with the initial condition $u(0) = 0$.

□

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