

DETERMINISTIC KPZ EQUATIONS WITH NONLOCAL “GRADIENT TERMS”

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ABSTRACT. The main goal of this paper is to prove existence and non-existence results for deterministic Kardar–Parisi–Zhang type equations involving non-local “gradient terms”. More precisely, let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\partial\Omega$ of class C^2 . For $s \in (0, 1)$, we consider problems of the form

$$\begin{cases} (-\Delta)^s u = \mu(x) |\mathbb{D}(u)|^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (\text{KPZ})$$

where $q > 1$ and $\lambda > 0$ are real parameters, f belongs to a suitable Lebesgue space, $\mu \in L^\infty(\Omega)$ and \mathbb{D} represents a nonlocal “gradient term”. Depending on the size of $\lambda > 0$, we derive existence and non-existence results. In particular, we solve several open problems posed in [4, Section 6] and [2, Section 7].

1. INTRODUCTION

In this paper we analyse the existence and non-existence of solutions for deterministic Kardar–Parisi–Zhang type equations involving non-local “gradient terms”. For $s, t \in (0, 1)$, we consider problems of the form

$$\begin{cases} (-\Delta)^s u = \mu(x) |\mathbb{D}_t(u)|^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (\text{KPZ})$$

depending on a real parameter $\lambda > 0$. Here, $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with boundary $\partial\Omega$ of class C^2 , f belongs to a suitable Lebesgue space, $\mu \in L^\infty(\Omega)$, $q \in (1, +\infty)$ and \mathbb{D}_t represents one of the following nonlocal “gradient terms” :

$$\begin{aligned} \circ \quad (-\Delta)^{\frac{t}{2}} u(x) &:= a_{N, \frac{t}{2}} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+t}} dy && \text{(Half } t\text{-Laplacian),} && (\text{KPZ}_1) \\ \circ \quad \nabla^t u(x) &:= \mu_{N, t} \int_{\mathbb{R}^N} \frac{(x - y)(u(x) - u(y))}{|x - y|^{N+t+1}} dy && \text{(Riesz } t\text{-Gradient),} && (\text{KPZ}_2) \\ \circ \quad \mathcal{D}_t u(x) &:= \left(\frac{a_{N, t}}{2} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2t}} dy \right)^{\frac{1}{2}} && \text{(Stein } t\text{-Functional).} && (\text{KPZ}_3) \end{aligned}$$

Note that the previous definitions make sense for any function $u \in C_c^\infty(\mathbb{R}^N)$. Also, let us point out that

$$a_{N, \sigma} := -\frac{2^{2\sigma} \Gamma\left(\frac{N}{2} + \sigma\right)}{\pi^{\frac{N}{2}} \Gamma(-\sigma)} \quad \text{and} \quad \mu_{N, \sigma} := \frac{2^\sigma \Gamma\left(\frac{N+\sigma+1}{2}\right)}{\pi^{\frac{N}{2}} \Gamma\left(\frac{1-\sigma}{2}\right)},$$

are normalization constants and “P.V.” stands for “in the principal value sense”. Since both these constants and the “principal value sense” will not play an important role in our work, we will omit them from now on.

Before going further, we would like to emphasize that the three different nonlocal “gradient terms” that we consider can be traced back many years ago. Since nowadays the *fractional Laplacian* does not need any further presentation, let us focus on the other two terms. As very well explained in [21, page 3], the origin of the *Riesz t -Gradient* seems to be [17]. Note also that this operator has been rediscovered several times since [17] and has received considerable attention in the last few years. See for instance [10, 14, 20, 24]. On the other hand, the *Stein t -Functional* can be at least traced back to [23]. Moreover, this operator naturally appears as the nonlocal equivalent to the gradient when considering the minimization of fractional Harmonic maps into the sphere. See for instance the recent papers [7, 15, 19].

In contrast with the local case

$$\begin{cases} -\Delta u = \mu(x) |\nabla u|^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where the literature is very extensive, there exist very few results dealing with equations of the form (KPZ). We refer to [2], by the first two authors, for a detailed introduction to the subject. However, since the publication of [2], some results have been published. For instance, we would like to mention the recent paper [1], where the authors establish existence and non-existence results for problems of the form (KPZ) with a local operator (the Laplacian instead of the fractional Laplacian) and a nonlocal nonlinearity. Also, let us mention [6], where the authors establish the equivalence between different notions of solution to problems of the form (KPZ).

In [2], the first two authors analyse the existence and non-existence of solutions to (KPZ), under the additional assumption $t = s \in (1/2, 1)$. The main goal of this paper is to refine the approach of [2] in order to deal also with the cases where $s \in (0, 1/2]$ and/or $t \neq s$. Depending on the real parameter $\lambda > 0$, we analyse the existence and non-existence of weak solutions to (KPZ) under the assumptions

$$\begin{cases} q \in (1, +\infty), \\ 0 < t < \min\{1, s(1 + (qN)^{-1})\}, \\ f \in L^m(\Omega) \text{ for some } m > N/s \text{ and } \mu \in L^\infty(\Omega). \end{cases} \quad (\text{A1})$$

Following [8, 9], we introduce the subsequent notion of weak solution to (KPZ) :

Definition 1.1. We say that u is a *weak solution* to (KPZ) if u and $|\mathbb{D}_t(u)|^q$ belong to $L^1(\Omega)$, $u \equiv 0$ (a.e.) in $\mathbb{R}^N \setminus \Omega$ and

$$\int_{\Omega} u(-\Delta)^s \phi \, dx = \int_{\Omega} (\mu(x)|\mathbb{D}_t(u)|^q + \lambda f(x)) \phi \, dx, \quad (1.1)$$

for all ϕ belonging to

$$\mathbb{X}^s(\Omega) := \left\{ \phi \in C^s(\mathbb{R}^N) : \phi(x) = 0 \text{ for all } x \in \mathbb{R}^N \setminus \Omega \text{ and } (-\Delta)^s \phi \in L^\infty(\Omega) \right\}.$$

Remark 1.2. With a slight abuse of notation we use (KPZ_{*i*}), $i = 1, 2, 3$, to refer to (KPZ) with $\mathbb{D}_t = (-\Delta)^{\frac{t}{2}}$, $\mathbb{D}_t = \nabla^t$ and $\mathbb{D}_t = \mathcal{D}_t$ respectively. See (KPZ₁), (KPZ₂) and (KPZ₃) for the corresponding definitions.

Our main existence result can be informally stated as follows:

Theorem 1.3. *Assume that (A1) holds true. Then, there exists $\lambda^* > 0$ such that, for all $0 < \lambda \leq \lambda^*$, (KPZ_{*i*}), $i = 1, 2, 3$, has a weak solution u . Moreover, $u \in W^{s,p}(\mathbb{R}^N) \cap C^{0,s}(\mathbb{R}^N)$ for all $1 < p < +\infty$.*

Theorem 1.3 is a particular case of the more general existence results proved in Section 3. We refer directly to Section 3 for more general statements. In particular, let us emphasize that, in Section 3, we substantially lower the regularity imposed on the data f . Furthermore, arguing as in the proof of [2, Theorem 1.3], it is possible to show that the regularity considered in Section 3 is almost optimal. Note also that our existence results solve several open problems posed in [2, 4].

The proofs of our existence results rely on the combination of fixed point arguments in the spirit of [13, 16] with *global* fractional Calderón–Zygmund regularity results for the fractional Poisson equation

$$\begin{cases} (-\Delta)^s u = h, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.2)$$

This approach was already implemented in [2] by the first two authors. However, the required *global* fractional Calderón–Zygmund regularity theory was not available for $s \in (0, 1/2]$. Furthermore, the Calderón–Zygmund regularity used in [2] (cf. [2, Section 3]) contains several imprecisions. In the recent paper [3], we establish sharp *global* fractional Calderón–Zygmund regularity results for (1.2) in the full range $s \in (0, 1)$ (fixing in particular the issues of [2, Section 3]). Having at hand these regularity results, the proofs of our existence results follow from a refinement of the fixed point approach implemented in [2]. We refer to Section 3 for more details.

Taking into account Theorem 1.3, it is very natural to ask whether the smallness assumption in λ is necessary or not. Note that, in [2, Theorem 1.2], the first two authors established the following non-existence result for (KPZ₃) with $t = s \in (0, 1)$ and $q = 2$.

Theorem 1.4. *Assume that (A1) holds true with $t = s$ and $q = 2$ and suppose that $\mu(x) \geq \mu_1 > 0$ and $f \geq 0$. Then, there exists $\lambda^{**} > 0$ such that, for all $\lambda > \lambda^{**}$, (KPZ₃) has no solution in $W^{s,2}(\mathbb{R}^N)$.*

For the sake of completeness, we prove here the non-existence results, below. The first theorem treats (KPZ₁) in case $q > \frac{2s+2}{t+2}$ and the second one regarding (KPZ₃) with no additional condition on q . These were left as open problems in [2, Section 7].

Theorem 1.5. *Assume that $s \in (0, 1)$, $t \in (0, 2s)$, $\mu(x) \geq \mu_1 > 0$ and $f \in L^1(\Omega)$ with $f \geq 0$. Then, for any $q > \frac{2s+2}{t+2}$ there exists $\lambda^{**} = \lambda^{**}(\mu_1, s, t, q, N, \Omega) > 0$ such that, for all $\lambda > \lambda^{**}$, (KPZ₁) has no solution in $W^{s,2}(\mathbb{R}^N)$.*

Theorem 1.6. *Assume that $s \in (0, 1)$, $t \in (0, 2s)$, $\mu(x) \geq \mu_1 > 0$ and $f \in L^1(\Omega)$ with $f \geq 0$. Then, for any $q > 1$ there exists $\lambda^{**} = \lambda^{**}(\mu_1, s, t, q, N, \Omega) > 0$ such that, for all $\lambda > \lambda^{**}$, (KPZ₃) has no solution u with $u \in W^{s,1}(\mathbb{R}^N)$ and $\mathcal{D}_t(u) \in L^q(\Omega)$.*

Remark 1.7. The non-existence for λ large in the case where $\mathbb{D}_t = \nabla^t$ remains completely open and a different approach is missing.

Organization of the paper. In the next section we introduce the main function spaces involved in our results and prove the continuity and compactness of the solution map for the fractional Poisson equation with L^1 -data. In Section 3, we prove our main existence results to (KPZ), from which Theorem 1.3 immediately follows. Finally, in Section 4, we prove Theorem 1.5 and Theorem 1.6 .

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2. FUNCTION SPACES AND TOOLS

We collect here the definition of the main function spaces involved in our results and some other tools. First of all, recall that, for all $s \in (0, 1)$ and $1 \leq p < +\infty$, the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy < +\infty \right\}.$$

It is a Banach space endowed with the usual norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|_{L^p(\mathbb{R}^N)}^p + \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}}.$$

Also, having at hand $W^{s,p}(\mathbb{R}^N)$, we define the space $W_0^{s,p}(\Omega)$ as

$$W_0^{s,p}(\Omega) := \{ u \in W^{s,p}(\mathbb{R}^N) : u \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega \},$$

and recall that, thanks to the Sobolev inequality, it is a Banach space endowed with the norm

$$\|u\|_{W_0^{s,p}(\Omega)} := \left(\iint_{D_\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{1/p}, \quad \text{where } D_\Omega := (\Omega \times \mathbb{R}^N) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega).$$

Next, we remind that, for any $s \in (0, 1)$ and $1 \leq p < +\infty$, the Bessel potential space is defined as

$$L^{s,p}(\mathbb{R}^N) := \overline{\{u \in C_c^\infty(\mathbb{R}^N)\}}^{1|_{L^{s,p}(\mathbb{R}^N)}},$$

where

$$\|u\|_{L^{s,p}(\mathbb{R}^N)} = \|(1 - \Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^N)} \quad \text{and} \quad (1 - \Delta)^{\frac{s}{2}} u = \mathcal{F}^{-1}((1 + |\cdot|^2)^{\frac{s}{2}} \mathcal{F}u), \quad \forall u \in C_c^\infty(\mathbb{R}^N).$$

Let us stress that, in the case where $s \in (0, 1)$ and $1 < p < +\infty$,

$$\|u\|_{L^{s,p}(\mathbb{R}^N)} := \|u\|_{L^p(\mathbb{R}^N)} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^N)}$$

is an equivalent norm for $L^{s,p}(\mathbb{R}^N)$ (see e.g. [23, Theorem 2]). By [23, Theorem 1.1], we also know that, if in addition $2N/(N + 2s) < p < +\infty$, then $L^{s,p}(\mathbb{R}^N)$ can be equipped with the equivalent norm

$$\|u\|_{L^{s,p}(\mathbb{R}^N)} := \|u\|_{L^p(\mathbb{R}^N)} + \|\mathcal{D}_s(u)\|_{L^p(\mathbb{R}^N)}.$$

In analogy with $W_0^{s,p}(\Omega)$, let us define

$$L_0^{s,p}(\Omega) := \{u \in L^{s,p}(\mathbb{R}^N) : u \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega\},$$

and stress that, if $0 < s < 1$ and $1 < p < +\infty$, it is a Banach space endowed with the norm

$$\|u\|_{L_0^{s,p}(\Omega)} := \|(-\Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^N)}.$$

If in addition $2N/(N + 2s) < p < +\infty$, then $L_0^{s,p}(\Omega)$ can also be equipped with the equivalent norm

$$\|u\|_{L_0^{s,p}(\Omega)} := \|\mathcal{D}_s(u)\|_{L^p(\mathbb{R}^N)}.$$

Let us as well recall that, for all $0 < \varepsilon < \sigma < 1$ and all $1 < p < +\infty$, by [5, Theorem 7.63, (g)], we have

$$L^{\sigma+\varepsilon,p}(\mathbb{R}^N) \subset W^{\sigma,p}(\mathbb{R}^N) \subset L^{\sigma-\varepsilon,p}(\mathbb{R}^N).$$

It is also well known that, for all $1 \leq p < +\infty$ and all $0 < \sigma \leq \sigma' < 1$,

$$W^{\sigma',p}(\mathbb{R}^N) \subset W^{\sigma,p}(\mathbb{R}^N),$$

and that (cf. [5, Theorem 7.63 (c)]), if in addition $1 < p < +\infty$,

$$L^{\sigma',p}(\mathbb{R}^N) \subset L^{\sigma,p}(\mathbb{R}^N).$$

Since the constants will be useful later on, let us emphasize there exists $\tilde{k} := \tilde{k}(\sigma, \sigma', p) \geq 1$ such that

$$\|u\|_{L^{\sigma,p}(\mathbb{R}^N)} \leq \tilde{k} \|u\|_{L^{\sigma',p}(\mathbb{R}^N)}, \quad \text{for all } u \in L^{\sigma',p}(\mathbb{R}^N), \quad (2.1)$$

and

$$\|u\|_{W^{\sigma,p}(\mathbb{R}^N)} \leq \tilde{k} \|u\|_{W^{\sigma',p}(\mathbb{R}^N)}, \quad \text{for all } u \in W^{\sigma',p}(\mathbb{R}^N), \quad (2.2)$$

The next result, we will use in the sequel, describes the regularity of the (unique) solution to (1.2). It is contained in [3], Theorems 1.3 and 5.2 and Corollaries 5.3–5.6.

Proposition 2.1. *Let $0 < s \leq t < \min\{1, 2s\}$ and let u be the unique solution of (1.2) with $h \in L^m(\Omega)$, $m \geq 1$. Then there exists $\tilde{C} = \tilde{C}(N, s, p, m, \Omega)$ such that*

$$\|u\|_{L^{t,p}(\mathbb{R}^N)} + \|u\|_{W^{t,p}(\mathbb{R}^N)} \leq \tilde{C} \|h\|_{L^m(\Omega)} \quad (2.3)$$

where $m_\star = \max\{1, \frac{N}{N(1-(t-s)^+)} + \varepsilon\}$, $\varepsilon > 0$ and ¹

$$\begin{cases} 1 \leq p < \frac{1}{(t-s)^+} & \text{for } m > \frac{N}{2s-t}, \\ 1 \leq p < \frac{Nm}{N-m_s+Nm(t-s)} & \text{for } m_\star \leq m < \frac{N}{2s-t}. \end{cases}$$

We conclude this section presenting a compactness result for the fractional Poisson equation that will be key in the proof of our main existence results. We keep here the notation introduced on [3, Page 3].

Proposition 2.2. *Let $0 < t < \min\{1, s(1+N^{-1})\}$ and $1 < p < N/(N(1+t-s)-s)$. The solution map*

$$\mathbb{G}_s : L^1(\Omega) \rightarrow L_0^{t,p}(\Omega), \quad f \mapsto \mathbb{G}_s[f] := \int_{\Omega} G_s(x, y) f(y) dy,$$

is well-defined, continuous and compact.

Proof. First of all, note that without loss of generality we can assume that $s \leq t < \min\{1, s(1+N^{-1})\}$. Indeed, once we have the result in this case, we can infer the result for $t \in (0, s)$ arguing as in the proof of [2, Proposition 3.10]. Also, let us stress that, by Proposition 2.1, the solution map \mathbb{G}_s is well-defined and continuous for all $1 < p < N/(N(1+t-s)-s)$. Hence, we just have to show that \mathbb{G}_s is compact for the same range of p .

Let $(f_n)_n \subset L^1(\Omega)$ be a sequence such that $\|f_n\|_{L^1(\Omega)} \leq 1$ for all $n \in \mathbb{N}$ and let $u_n = \mathbb{G}_s(f_n)$ for all $n \in \mathbb{N}$. By [8, Proposition 2.6] we know that, up to a subsequence, $(u_n)_n$ is strongly convergent in $L^q(\Omega)$ for all $1 \leq q < N/(N-2s)$. Moreover, combining this strong convergence with Vitali's convergence theorem, we get that $(u_n/\delta^s)_n$ is strongly convergent in $L^r(\Omega)$ for all $1 \leq r < N/(N-s)$. Having at hand the strong convergence of $(u_n)_n$ in $L^q(\Omega)$ for all $1 \leq q < N/(N-2s)$, to end the proof, we just have to show that $((-\Delta)^{\frac{t}{2}} u_n)_n$ is strongly convergent in $L^p(\mathbb{R}^N)$ for all $1 < p < N/(N(1+t-s)-s)$.

For $0 < \alpha < 2$, we consider the integral operator

$$T_\alpha : L^1(\Omega) \rightarrow L^\gamma(\Omega), \quad f \mapsto \int_{\mathbb{R}^N} \frac{f(y) \mathbf{1}_\Omega(y)}{|x-y|^{N-\alpha}} dy.$$

and note that, by [12, Lemma 7.12], T_α is well-defined and continuous for all $1 \leq \gamma < N/(N-\alpha)$. Moreover, following step by step the proof of [22, Theorem 2.2], one can prove that T_α is compact for all $1 \leq \gamma < N/(N-\alpha)$. Combining that T_α is compact for all $1 \leq \gamma < N/(N-\alpha)$, [3, Lemma 5.1] and Hölder inequality, it is immediate to see that $((-\Delta)^{\frac{t}{2}} u_n)_n$ is strongly convergent in $L^p(\Omega)$ for all $1 \leq p < N/(N(1+t-s)-s)$.

We finally deal with the convergence in $L^p(\mathbb{R}^N \setminus \Omega)$. We consider separately two regions. For $R := \frac{1}{3} + \frac{4}{3}(\text{diam}(\Omega) + \text{dist}(0, \Omega))$, the strong convergence of $((-\Delta)^{\frac{t}{2}} u_n)_n$ in $L^q(\mathbb{R}^N \setminus B_R(0))$ for all $1 \leq p < N/(N-2s)$ immediately follows from [3, (5.8)] and the strong convergence of $(u_n)_n$ in $L^q(\Omega)$. On the other hand, the strong convergence of $((-\Delta)^{\frac{t}{2}} u_n)_n$ in $L^p(B_R(0) \setminus \Omega)$ for all $1 \leq p < N/(N(1+t-s)-s)$ follows from the strong convergence of $(u_n/\delta^s)_n$ for $1 \leq r < N/(N-s)$ arguing as in the proof of Proposition 2.1. \square

Corollary 2.3. *Let $0 < t < \min\{1, s(1+N^{-1})\}$ and $1 < p < N/(N(1+t-s)-s)$. The solution map*

$$\mathbb{G}_s : L^1(\Omega) \rightarrow W_0^{t,p}(\Omega), \quad f \mapsto \mathbb{G}_s[f] := \int_{\Omega} G_s(x, y) f(y) dy,$$

is well-defined, continuous and compact.

Proof. Having at hand Proposition 2.2 the result follows arguing as in the proof of [3, Corollary 5.6] \square

Remark 2.4.

- In the case where $1/2 < t < \min\{1, s(1+N^{-1})\}$, Proposition 2.2 and Corollary 2.3 can be proved arguing as in the proof of [2, Proposition 3.10].
- We believe Proposition 2.2 and Corollary 2.3 are of independent interest and will be useful elsewhere.

¹We use, here and in the sequel, the convention $1/a^+ = +\infty$ if $a \leq 0$.

3. EXISTENCE RESULTS

This section is devoted to prove existence results for (KPZ) with the different choices of \mathbb{D}_t present in the introduction. We will analyse in parallel $\mathbb{D}_t = (-\Delta)^{\frac{s}{2}}$ and $\mathbb{D}_t = \nabla^t$ and separately $\mathbb{D}_t = \mathcal{D}_t$. Let us emphasize that the main existence result stated in the introduction, namely Theorem 1.3, immediately follows from the results of this section.

We first analyse the case $\mathbb{D}_t = (-\Delta)^{\frac{s}{2}}$ (the case $\mathbb{D}_t = \nabla^t$ follows arguing on the exact same way, as we will detail later on). More precisely, for $q \in (1, +\infty)$, we analyse the existence of weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)|(-\Delta)^{\frac{s}{2}} u|^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (\text{KPZ}_1)$$

Let us impose $0 < t < \min\{1, s(1 + N^{-1})\}$ and set

$$\bar{q}(m, s, t) := \begin{cases} +\infty, & \text{if } t \leq s \text{ and } m \geq N/s, \\ s/(N(t-s)), & \text{if } t > s \text{ and } m > N/s, \\ N/(N-ms), & \text{if } t \leq s \text{ and } 1 \leq m < N/s, \\ N/(N-sm+mN(t-s)), & \text{if } t > s \text{ and } 1 \leq m \leq N/s. \end{cases} \quad (3.1)$$

Having at hand $\bar{q} \in (1, +\infty]$, our main result concerning (KPZ₁) reads as follows:

Theorem 3.1. *Assume that $0 < t < \min\{1, s(1 + N^{-1})\}$, $f \in L^m(\Omega)$ for some $m \geq 1$ and $\mu \in L^\infty(\Omega)$. Then, for all $1 < q < \bar{q}$, there exists $\lambda_\star > 0$ such that, for all $0 < \lambda \leq \lambda_\star$, (KPZ₁) has a weak solution u . Moreover:*

- If $m \geq N/s$, then $u \in W^{s,p}(\mathbb{R}^N) \cap C^{0,s}(\mathbb{R}^N)$ for all $1 < p < +\infty$.
- If $1 \leq m < N/s$, then $u \in W^{s,p}(\mathbb{R}^N)$ for all $1 < p < mN/(N-ms)$.

Proof of Theorem 3.1. We use some ideas of [2, Sections 4 and 6] and consider separately the cases $m > N/s$ and $1 \leq m \leq N/s$.

Case 1: $m > N/s$.

First of all, observe that, without loss of generality, we can assume that $N/s < m < 1/(q(t-s))^+$. Then, let us fix $r = r(m, s, t, q) > 0$ such that $1 < qm < r < \frac{1}{(t-s)^+}$ and define

$$\lambda_\star := \frac{q-1}{q\|f\|_{L^m(\Omega)}} \left(q(\tilde{C}\tilde{k})^q \|\mu\|_{L^\infty(\Omega)} |\Omega|^{\frac{r-qm}{mr}} \right)^{-\frac{1}{q-1}}, \quad (3.2)$$

with $\tilde{C} > 0$ as in Proposition 2.1 and \tilde{k} as in (2.1). Since $q > 1$, we know (cf. [2, Lemma 4.1]) there exists a unique $\ell \in (0, \infty)$ such that

$$\tilde{C} \left(\|\mu\|_{L^\infty(\Omega)} |\Omega|^{\frac{r-qm}{mr}} \tilde{k}^q \ell + \lambda_\star \|f\|_{L^m(\Omega)} \right) = \ell^{\frac{1}{q}}. \quad (3.3)$$

Having at hand λ_\star and ℓ , we define

$$E_\eta := \left\{ v \in L_0^{\gamma, 1+\eta}(\Omega) : \|v\|_{L_0^{\gamma, r}(\Omega)} \leq \ell^{\frac{1}{q}} \right\}, \quad (3.4)$$

with

$$\gamma := \max\{t, s\}, \quad \text{and } 0 < \eta < \min \left\{ q-1, \frac{s-N(\gamma-s)}{N(1+(\gamma-s))-s} \right\}, \quad (3.5)$$

and point out that E_η is a closed, bounded and convex subset of $L_0^{\gamma, 1+\eta}(\Omega)$ (the fact that it is bounded can be checked using the standard fractional Sobolev embeddings and interpolation). In order to prove the existence of a weak solution to (KPZ₁) belonging to E_η we use Schauder's fixed point Theorem. Let us consider

$$T_1 : E_\eta \rightarrow L_0^{\gamma, 1+\eta}(\Omega), \quad \varphi \mapsto u, \quad (3.6)$$

where u is the unique weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)|(-\Delta)^{\frac{s}{2}} \varphi|^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (3.7)$$

and observe that, if we prove that T_1 has a fixed point in E_η , the existence part immediately follows. Note that, by Proposition 2.2, the operator T_1 is well defined. Hence, to end the proof in this case, we just have to prove that T_1 is continuous and compact and that $T_1(E_\eta) \subset E_\eta$.

We start by proving that $T_1(E_\eta) \subset E_\eta$. Let $\varphi \in E_\eta$ and $u = T_1(\varphi)$. Using Proposition 2.1, it is immediate to see that $u \in L_0^{\gamma, 1+\eta}(\Omega)$ and that

$$\begin{aligned} \|u\|_{L_0^{\gamma, r}(\Omega)} &\leq \tilde{C} \left(\lambda_* \|f\|_{L^m(\Omega)} + \|\mu\|_{L^\infty(\Omega)} \| |(-\Delta)^{\frac{t}{2}} \varphi|^q \|_{L^m(\Omega)} \right) \\ &\leq \tilde{C} \left(\lambda_* \|f\|_{L^m(\Omega)} + \|\mu\|_{L^\infty(\Omega)} |\Omega|^{\frac{r-qm}{rm}} \| |(-\Delta)^{\frac{t}{2}} \varphi \|^q_{L^r(\Omega)} \right) \\ &\leq \tilde{C} \left(\lambda_* \|f\|_{L^m(\Omega)} + \|\mu\|_{L^\infty(\Omega)} |\Omega|^{\frac{r-qm}{rm}} \|\varphi\|_{L_0^{t, r}(\Omega)}^q \right) \\ &\leq \tilde{C} \left(\lambda_* \|f\|_{L^m(\Omega)} + \|\mu\|_{L^\infty(\Omega)} |\Omega|^{\frac{r-qm}{rm}} \tilde{k}^q \|\varphi\|_{L_0^{\gamma, r}(\Omega)}^q \right) \leq \ell^{\frac{1}{q}}. \end{aligned} \quad (3.8)$$

Hence, it follows that $T_1(E_\eta) \subset E_\eta$.

Next, we prove that T_1 is compact. Let $(\varphi_n)_n \subset E_\eta$ be such that $\|\varphi_n\|_{L_0^{\gamma, 1+\eta}(\Omega)} \leq 1$ for all $n \in \mathbb{N}$. Also, let $h_n := \mu(x) |(-\Delta)^{\frac{t}{2}} \varphi_n|^q + \lambda f(x)$ for all $n \in \mathbb{N}$. Arguing as in (3.8), it is immediate to check that $(h_n)_n$ is bounded in $L^1(\Omega)$ and thus, the compactness of T_1 immediately follows from Proposition 2.2.

Finally, we prove that T_1 is continuous. Let $(\varphi_n)_n \subset E_\eta$ be a sequence such that $\varphi_n \rightarrow \varphi$ in $L_0^{\gamma, 1+\eta}(\Omega)$ and let $u_n = T_1(\varphi_n)$ for all $n \in \mathbb{N}$ and $u = T_1(\varphi)$. Note that

$$\begin{cases} (-\Delta)^s(u_n - u) = \mu(x) \left(|(-\Delta)^{\frac{t}{2}} \varphi_n|^q - |(-\Delta)^{\frac{t}{2}} \varphi|^q \right), & \text{in } \Omega, \\ u_n - u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (3.9)$$

If we show that the L^1 -norm of the right hand side in the (3.9) goes to 0 as $n \rightarrow \infty$, the continuity of T_1 immediately follows from Proposition 2.2 and the existence in the case where $m > N/s$ will follow. By direct computations (using Hölder inequality and the Mean value Theorem), it follows that

$$\left\| \mu(x) \left(|(-\Delta)^{\frac{t}{2}} \varphi_n|^q - |(-\Delta)^{\frac{t}{2}} \varphi|^q \right) \right\|_{L^1(\Omega)} \leq C \|(-\Delta)^{\frac{t}{2}}(\varphi_n - \varphi)\|_{L^q(\Omega)}, \quad (3.10)$$

for some $C > 0$ depending only on $q, r, \|\mu\|_{L^\infty(\Omega)}, |\Omega|$ and ℓ . On the other hand, using (2.1) and Littlewood's inequality (or interpolation in L^p -spaces), we infer that

$$\begin{aligned} \|(-\Delta)^{\frac{t}{2}}(\varphi_n - \varphi)\|_{L^q(\Omega)} &\leq \|\varphi_n - \varphi\|_{L_0^{t, q}(\Omega)} \leq \tilde{k} \|\varphi_n - \varphi\|_{L_0^{\gamma, q}(\Omega)} = \tilde{k} \|(-\Delta)^{\frac{\gamma}{2}}(\varphi_n - \varphi)\|_{L^q(\mathbb{R}^N)} \\ &\leq \tilde{k} \|(-\Delta)^{\frac{\gamma}{2}}(\varphi_n - \varphi)\|_{L^{1+\eta}(\mathbb{R}^N)}^\theta \|(-\Delta)^{\frac{\gamma}{2}}(\varphi_n - \varphi)\|_{L^r(\mathbb{R}^N)}^{1-\theta} = \tilde{k} \|\varphi_n - \varphi\|_{L_0^{\gamma, 1+\eta}(\Omega)}^\theta \|\varphi_n - \varphi\|_{L_0^{\gamma, r}(\Omega)}^{1-\theta} \\ &\leq (2\ell^{\frac{1}{q}})^{1-\theta} \tilde{k} \|\varphi_n - \varphi\|_{L_0^{\gamma, 1+\eta}(\Omega)}, \end{aligned} \quad (3.11)$$

with $\frac{1}{q} = \frac{\theta}{1+\eta} + \frac{1-\theta}{r}$. Since $\varphi_n \rightarrow \varphi$ in $L_0^{\gamma, 1+\eta}(\Omega)$, combining (3.10) and (3.11), we conclude that

$$\lim_{n \rightarrow \infty} \left\| \mu \left(|(-\Delta)^{\frac{t}{2}} \varphi_n|^q - |(-\Delta)^{\frac{t}{2}} \varphi|^q \right) \right\|_{L^1(\Omega)} = 0,$$

as desired. The proof of the existence in the case where $m > N/s$ is thus finished. Once we have the existence of a weak solution $u \in E_\eta$, the claimed regularity immediately follows from the definition of E_η , our choice of r , Proposition 2.1 and [18, Proposition 1.4 (iii)].

Case 2: $1 \leq m \leq N/s$.

First of all, let us fix $r = r(m, s, t, q) > 0$ such that $1 < qm < r < \frac{mN}{(N-ms+mN(\gamma-s))^+}$ and consider λ_* and ℓ as in (3.2) and (3.3) respectively. Then, we define

$$\tilde{E}_\eta := \left\{ v \in L_0^{\gamma, 1+\eta}(\Omega) : \|v\|_{L_0^{\gamma, 1+\eta}(\Omega)} \leq M \text{ and } \|v\|_{L_0^{\gamma, r}(\Omega)} \leq \ell^{\frac{1}{q}} \right\}, \quad (3.12)$$

with γ and η as in (3.5) and

$$M := \tilde{C}_1 \left(\|\mu\|_{L^\infty(\Omega)} |\Omega|^{\frac{r-q}{r}} \tilde{k}^q \ell + \lambda_* \|f\|_{L^1(\Omega)} \right),$$

where $\tilde{C}_1 > 0$ is the constant that appears in Proposition 2.1 for $m = 1$. It is immediate to see that \tilde{E}_η is a bounded, closed and convex set of $L_0^{\gamma, 1+\eta}(\Omega)$. Moreover, arguing exactly as in the first case, one can prove that

$$\tilde{T}_1 : \tilde{E}_\eta \rightarrow L_0^{\gamma, 1+\eta}(\Omega), \quad \varphi \mapsto u, \quad (3.13)$$

where u is the unique weak solution to (3.7), is well-defined, continuous, compact and satisfies $\tilde{T}_1(\tilde{E}_\eta) \subset \tilde{E}_\eta$. Hence, applying again Schauder's fixed point Theorem, the existence follows also in this case. Having at hand the existence, the claimed regularity follows again from Proposition 2.1. \square

Next, we analyse the existence of weak solution to (KPZ) in the case where $\mathbb{D}_t = \nabla^t$. More precisely, we analyse the existence of weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)|\nabla^t u|^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (\text{KPZ}_2)$$

Our main result concerning (KPZ₂) can be formulated as follows:

Theorem 3.2. *Assume that $0 < t < \min\{1, s(1 + N^{-1})\}$, $f \in L^m(\Omega)$ for some $m \geq 1$ and $\mu \in L^\infty(\Omega)$. Then, for all $1 < q < \bar{q}$, there exists $\lambda_\star > 0$ such that, for all $0 < \lambda \leq \lambda_\star$, (KPZ₂) has a weak solution u . Moreover:*

- *If $m \geq N/s$, then $u \in W^{s,p}(\mathbb{R}^N) \cap C^{0,s}(\mathbb{R}^N)$ for all $1 < p < +\infty$.*
- *If $1 \leq m < N/s$, then $u \in W^{s,p}(\mathbb{R}^N)$ for all $1 < p < mN/(N - ms)$.*

Remark 3.3. Let us emphasize that, having at hand [21, Theorem 1.7], the proof of Theorem 3.2 follows arguing exactly as in the proof of Theorem 3.1.

Finally, we analyse the existence of weak solution in the slightly more involved case where $\mathbb{D}_t = \mathcal{D}_t$. More precisely, we analyse the existence of weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)(\mathcal{D}_t u)^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (\text{KPZ}_3)$$

Setting $\gamma := \max\{t, s\}$ and

$$\bar{m}(s, t) := \frac{2N}{N + 2s - 2N(t - s)_+}$$

our main result concerning (KPZ₃) reads as follows:

Theorem 3.4. *Assume that $0 < t < \min\{1, s(1 + N^{-1})\}$, $f \in L^m(\Omega)$ for some $m > \bar{m}$ and $\mu \in L^\infty(\Omega)$. Then, for all $1 < q < \bar{q}$, there exists $\lambda_\star > 0$ such that, for all $0 < \lambda \leq \lambda_\star$, (KPZ₃) has a weak solution u . Moreover:*

- *If $m \geq N/s$, then $u \in W^{s,p}(\mathbb{R}^N) \cap C^{0,s}(\mathbb{R}^N)$ for all $1 < p < +\infty$.*
- *If $\bar{m} < m < N/s$, then $u \in W^{s,p}(\mathbb{R}^N)$ for all $1 < p < mN/(N - ms)$.*

Proof. We consider separately the cases $m > N/s$ and $\bar{m} < m \leq N/s$.

Case 1: $m > N/s$.

First of all, observe that, without loss of generality, we can assume that $N/s < m < 1/(q(t - s)^+)$. Then, let us fix $r = r(m, s, t, q) > 0$ such that $\max\{qm, 2\} < r < \frac{1}{(t-s)_+}$ and define

$$\lambda_\star := \frac{q-1}{q\|f\|_{L^m(\Omega)}} \left(q(\tilde{C}\tilde{k})^q \|\mu\|_{L^\infty(\Omega)} |\Omega|^{\frac{r-qm}{mr}} \right)^{-\frac{1}{q-1}}, \quad (3.14)$$

with $\tilde{C} > 0$ as in Proposition 2.1 and \tilde{k} as in (2.1). Since $q > 1$, we know (cf. [2, Lemma 4.1]) there exists a unique $\ell \in (0, \infty)$ such that

$$\tilde{C} \left(\|\mu\|_{L^\infty(\Omega)} |\Omega|^{\frac{r-qm}{mr}} \tilde{k}^q \ell + \lambda_\star \|f\|_{L^m(\Omega)} \right) = \ell^{\frac{1}{q}}. \quad (3.15)$$

Having at hand λ_\star and ℓ , we define

$$E_\eta := \left\{ v \in L_0^{\gamma, 1+\eta}(\Omega) : \|v\|_{L_0^{\gamma, r}(\Omega)} \leq \ell^{\frac{1}{q}} \right\}, \quad (3.16)$$

with

$$\gamma = \max\{t, s\} \quad \text{and} \quad 0 < \eta < \min \left\{ q-1, \frac{s - N(\gamma - s)}{N(1 + (\gamma - s)) - s} \right\}, \quad (3.17)$$

and we point out that E_η is a closed, bounded and convex subset of $L_0^{\gamma, 1+\eta}(\Omega)$. To prove the existence of a weak solution to (KPZ₃) belonging to E_η , we use again Schauder's fixed point Theorem. Let us define

$$T_2 : E_\eta \rightarrow L_0^{\gamma, 1+\eta}(\Omega), \quad \varphi \mapsto u, \quad (3.18)$$

where u the unique weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)(\mathcal{D}_t \varphi)^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (3.19)$$

Note that, by Proposition 2.2, the operator T_2 is well defined. Hence, to conclude the proof in this case, we just have to prove that T_2 is continuous and compact and that $T_2(E_\eta) \subset E_\eta$. The compactness of T_2 and

the fact that $T_2(E_\eta) \subset E_\eta$ can be proved arguing exactly as in the proof of Theorem 3.1. However, to prove that T_2 is continuous we have to argue in a different way. Let $(\varphi_n)_n \subset E_\eta$ be a sequence such that $\varphi_n \rightarrow \varphi$ in $L_0^{\gamma,1+\eta}(\Omega)$ and let $u_n = T_2(\varphi_n)$ for all $n \in \mathbb{N}$ and $u = T_2(\varphi)$. Note that

$$\begin{cases} (-\Delta)^s(u_n - u) = \mu(x) \left((\mathcal{D}_t \varphi_n)^q - (\mathcal{D}_t \varphi)^q \right), & \text{in } \Omega, \\ u_n - u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (3.20)$$

If we show that the L^1 -norm of the right hand side in (3.20) goes to 0 as $n \rightarrow \infty$, the continuity of T_2 immediately follows from Proposition 2.2. We actually prove something more general from which the existence part of the result in the case $m > N/s$ immediately follows.

Claim: For all $1 < \alpha < r$, it follows that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |(\mathcal{D}_t \varphi_n(x))^\alpha - (\mathcal{D}_t \varphi(x))^\alpha| dx = 0. \quad (3.21)$$

Proof of the claim. First of all, let $2 \leq \beta < r$ be fixed but arbitrary. Using the Mean value Theorem and Hölder and triangular inequalities, one can easily get that

$$\begin{aligned} \int_{\Omega} |(\mathcal{D}_t \varphi_n(x))^\beta - (\mathcal{D}_t \varphi(x))^\beta| dx &= \int_{\Omega} |((\mathcal{D}_t \varphi_n(x))^2)^{\frac{\beta}{2}} - ((\mathcal{D}_t \varphi(x))^2)^{\frac{\beta}{2}}| dx \\ &\leq \frac{\beta}{2} \int_{\Omega} |(\mathcal{D}_t \varphi_n(x))^2 - (\mathcal{D}_t \varphi(x))^2| ((\mathcal{D}_t \varphi_n(x))^2 + (\mathcal{D}_t \varphi(x))^2)^{\frac{\beta-2}{2}} dx \\ &\leq \frac{\beta}{2} \int_{\Omega} (\mathcal{D}_t(\varphi_n - \varphi)(x)) (\mathcal{D}_t(\varphi_n + \varphi)(x)) ((\mathcal{D}_t \varphi_n(x))^2 + (\mathcal{D}_t \varphi(x))^2)^{\frac{\beta-2}{2}} dx \\ &\leq \frac{\sqrt{2}}{2} \beta \int_{\Omega} (\mathcal{D}_t(\varphi_n - \varphi)(x)) (\mathcal{D}_t \varphi_n(x) + \mathcal{D}_t \varphi(x))^{\beta-1} dx \leq \frac{\sqrt{2}}{2} \beta \|\mathcal{D}_t(\varphi_n - \varphi)\|_{L^\beta(\Omega)} \|\mathcal{D}_t \varphi_n + \mathcal{D}_t \varphi\|_{L^\beta(\Omega)}^{\beta-1} \\ &\leq C (\|\mathcal{D}_t \varphi_n\|_{L^\beta(\Omega)}^{\beta-1} + \|\mathcal{D}_t \varphi\|_{L^\beta(\Omega)}^{\beta-1}) \|\mathcal{D}_t(\varphi_n - \varphi)\|_{L^\beta(\Omega)} \leq C (\|\varphi_n\|_{L_0^{t,\beta}(\Omega)}^{\beta-1} + \|\varphi\|_{L_0^{t,\beta}(\Omega)}^{\beta-1}) \|\varphi_n - \varphi\|_{L_0^{t,\beta}(\Omega)}, \end{aligned}$$

for some $C > 0$ depending only on β . Then, arguing exactly as in the proof of (3.11), we conclude that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |(\mathcal{D}_t \varphi_n(x))^\beta - (\mathcal{D}_t \varphi(x))^\beta| dx = 0.$$

Since $\beta \in [2, r)$ was fixed but arbitrary, we have proved (3.21) for all $2 \leq \alpha < r$.

Hence, it remains to deal with the case $1 < \alpha < 2$. To that end, note that $(\mathcal{D}_t \varphi_n)_n \subset L^2(\Omega)$ is a bounded non-negative sequence such that $\|\mathcal{D}_t \varphi_n\|_{L^2(\Omega)} \rightarrow \|\mathcal{D}_t \varphi\|_{L^2(\Omega)}$ as $n \rightarrow \infty$. Hence, it follows that $\mathcal{D}_t \varphi_n \rightarrow \mathcal{D}_t \varphi$ in $L^2(\Omega)$ as $n \rightarrow \infty$. Also, observe that, for any $\alpha \in (1, 2)$,

$$\begin{aligned} \int_{\Omega} |(\mathcal{D}_t \varphi_n(x))^\alpha - (\mathcal{D}_t \varphi(x))^\alpha| dx \\ \leq C \|\mathcal{D}_t(\varphi_n) - \mathcal{D}_t(\varphi)\|_{L^\alpha(\Omega)} (\|\mathcal{D}_t \varphi_n\|_{L^\alpha(\Omega)}^{\alpha-1} + \|\mathcal{D}_t \varphi\|_{L^\alpha(\Omega)}^{\alpha-1}) \leq \tilde{C} \|\mathcal{D}_t \varphi_n - \mathcal{D}_t \varphi\|_{L^2(\Omega)}, \end{aligned}$$

with $C > 0$ depending only on $\alpha > 0$ and $\tilde{C} > 0$ depending only on $\alpha, r, |\Omega|$ and ℓ . Combining this chain of inequalities with the fact that $\mathcal{D}_t \varphi_n \rightarrow \mathcal{D}_t \varphi$ in $L^2(\Omega)$ as $n \rightarrow \infty$ we conclude that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |(\mathcal{D}_t \varphi_n(x))^\alpha - (\mathcal{D}_t \varphi(x))^\alpha| dx = 0.$$

To conclude the proof in the case where $m > N/s$ it just remains to prove the claimed regularity, that follows thanks to Proposition 2.1 and the definition of E_η .

Case 2: $\bar{m} < m \leq N/s$.

First of all, let us fix $r = r(m, s, t, q) > 0$ such that $\max\{2, qm\} < r < \frac{mN}{(N-ms+mN(\gamma-s))^+}$ and consider λ_* and ℓ as in (3.14) and (3.15) respectively. Then, let us define

$$\tilde{E}_\eta := \left\{ v \in L_0^{\gamma,1+\eta}(\Omega) : \|v\|_{L_0^{\gamma,1+\eta}(\Omega)} \leq M \text{ and } \|v\|_{L_0^{\gamma,r}(\Omega)} \leq \ell^{\frac{1}{q}} \right\}, \quad (3.22)$$

with γ and η as in (3.17) and

$$M := C \left(\|\mu\|_{L^\infty(\Omega)} |\Omega|^{\frac{r-q}{r}} \tilde{k}^q \ell + \lambda_* \|f\|_{L^1(\Omega)} \right),$$

with $C > 0$ depending only on Ω, N, s and t (cf. Proposition 2.1). It is easy to check that \tilde{E}_η is a bounded, closed and convex set of $L_0^{\gamma,1+\eta}(\Omega)$. Moreover, arguing as we did in the first case, one can prove that

$$\tilde{T}_2 : \tilde{E}_\eta \rightarrow L_0^{\gamma,1+\eta}(\Omega), \quad \varphi \mapsto u, \quad (3.23)$$

where u is the unique weak solution to (3.19), is well-defined, continuous, compact and satisfies $\widetilde{T}_2(\widetilde{E}_\eta) \subset \widetilde{E}_\eta$. Hence, applying again Schauder's fixed point Theorem, the existence follows also in this case. Having at hand the existence, the claimed regularity follows again from Proposition 2.1. \square

Remark 3.5. As already mentioned in the introduction, our existence results solve several open problems posed in [2, 4]. Let us also emphasize that our proofs here are substantially shorter and simpler than the corresponding ones in [2].

4. NON-EXISTENCE RESULTS

This section is devoted to the proofs of the nonexistence results. We start by proving Theorem 1.5. As already mentioned in the introduction, the proof of this result is completely different from its counterpart for (KPZ₃), (i.e. [2, Theorem 1.2]).

Proof of Theorem 1.5. Let $u \in W_0^{s,2}(\Omega)$ be a weak solution to (KPZ₁), i.e. it satisfies

$$\int_{\Omega} (-\Delta)^s u(x) v(x) \geq \int_{\Omega} \mu(x) |(-\Delta)^{\frac{s}{2}} u|^q v(x) dx + \lambda \int_{\Omega} f v(x) dx,$$

for any $v \in \mathbb{X}^s(\Omega)$.

Let moreover $\phi \in W_0^{s,2}(\Omega) \cap C^s(\mathbb{R}^N)$ be the unique solution to

$$\begin{cases} (-\Delta)^s \phi = 1, & \text{in } \Omega, \\ \phi = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

First of all, note that $\phi \in \mathbb{X}^s(\Omega)$, so that we can use ϕ as test function in (KPZ₁) and get that

$$\int_{\Omega} u dx = \int_{\Omega} \mu(x) |(-\Delta)^{\frac{s}{2}} u|^q \phi dx + \lambda \int_{\Omega} f \phi dx \geq \mu_1 \int_{\Omega} |(-\Delta)^{\frac{s}{2}} u|^q \phi dx + \lambda \int_{\Omega} f \phi dx. \quad (4.1)$$

On the other hand, let $\psi \in W_0^{\frac{s}{2},2}(\Omega) \cap C^{\frac{s}{2}}(\mathbb{R}^N)$ be the unique solution to the problem

$$\begin{cases} (-\Delta)^{\frac{s}{2}} \psi = 1, & \text{in } \Omega, \\ \psi = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (4.2)$$

Since $W_0^{s,2}(\Omega) \subset W_0^{\frac{s}{2},2}(\Omega)$ we can test (4.2) with u and integrate by parts, so that we obtain using (4.1):

$$\int_{\Omega} \psi (-\Delta)^{\frac{s}{2}} u dx = \int_{\Omega} u dx \geq \mu_1 \int_{\Omega} |(-\Delta)^{\frac{s}{2}} u|^q \phi dx + \lambda \int_{\Omega} f \phi dx. \quad (4.3)$$

Moreover, by Young inequality we easily get that

$$\int_{\Omega} \psi (-\Delta)^{\frac{s}{2}} u dx \leq \mu_1 \int_{\Omega} |(-\Delta)^{\frac{s}{2}} u|^q \phi dx + c_q \mu_1^{-\frac{1}{q-1}} \int_{\Omega} \frac{\psi^{q'}}{\phi^{\frac{q'}{q}}} dx. \quad (4.4)$$

Thus, combining (4.3) and (4.4), we obtain that

$$\lambda \int_{\Omega} f \phi dx \leq c_q \mu_1^{-\frac{1}{q-1}} \int_{\Omega} \frac{\psi^{q'}}{\phi^{\frac{q'}{q}}} dx. \quad (4.5)$$

Finally, note that (see e.g. [11, Eq. (1.15)]) there exists $c_0 > 0$ (depending only on Ω , s and N) such that

$$c_0^{-1} \delta^s(x) \leq \phi(x) \leq c_0 \delta^s(x) \quad \text{and} \quad c_0^{-1} \delta^{\frac{s}{2}}(x) \leq \psi(x) \leq c_0 \delta^{\frac{s}{2}}(x), \quad \text{in } \overline{\Omega}. \quad (4.6)$$

Hence, since $q > \frac{2s+2}{s+2}$ the right hand side in (4.5) is bounded, and thus necessarily

$$\lambda \leq c_0 \frac{c_q \mu_1^{-\frac{1}{q-1}} \int_{\Omega} \frac{\psi^{q'}}{\phi^{\frac{q'}{q}}} dx}{\int_{\Omega} f(x) \delta^s(x) dx} := \lambda^{**}. \quad (4.7)$$

Thus we get a contradiction if λ is chosen large enough. \square

Remark 4.1. Observe that the bound that appears in Theorem 1.5, on the power q seems to be technical, but we are not able to weaken.

Next we give the proof the Theorem 1.6 which is a slightly generalization to Theorem 1.4.

Proof of Theorem 1.6. Let $u \in W^{s,1}(\mathbb{R}^N)$ be a weak solution to problem (KPZ₃) with $\mathcal{D}_t(u) \in L^q(\Omega)$. Consider $\phi \in \mathbb{X}^s(\Omega)$ to be a nonnegative function such that

$$\int_{\Omega} f(x)\phi^{q'}(x)dx > 0.$$

Using $\phi^{q'}$ as test function in (KPZ₃), it holds that

$$\int_{\Omega} u(x)(-\Delta)^s\phi^{q'}(x)dx \geq \mu_1 \int_{\Omega} (\mathcal{D}_t(u)(x))^q\phi^{q'}(x)dx + \lambda \int_{\Omega} f(x)\phi^{q'}(x)dx.$$

Using that, for any $a, b \geq 0, \alpha > 1$, there exists $c_{\alpha} > 0$ such that

$$|a^{\alpha} - b^{\alpha}| \geq c_{\alpha}|a - b|(a^{m-1} + b^{m-1}),$$

and since $u \in W^{s,1}(\mathbb{R}^N)$ with $\mathcal{D}_t(u) \in L^q(\Omega)$, it holds that

$$\begin{aligned} \int_{\Omega} u(x)(-\Delta)^s\phi^{q'}(x)dx &= \iint_{D_{\Omega}} \frac{(u(x) - u(y))(\phi^{q'}(x) - \phi^{q'}(y))}{|x - y|^{N+2s}} dydx \\ &\leq c_{\alpha} \iint_{D_{\Omega}} \frac{|u(x) - u(y)||\phi(x) - \phi(y)|}{|x - y|^{N+2s}} (\phi^{q'-1}(x) + \phi^{q'-1}(y)) dydx \\ &\leq 2c_{\alpha} \iint_{D_{\Omega}} \frac{|u(x) - u(y)||\phi(x) - \phi(y)|}{|x - y|^{N+2s}} \phi^{q'-1}(x) dydx. \end{aligned}$$

Thus, using Hölder and Young inequalities, it holds that

$$\begin{aligned} \int_{\Omega} u(x)(-\Delta)^s\phi^{q'}(x)dx &\leq 2c_{\alpha} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2t}} dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2(2s-t)}} dy \right)^{\frac{1}{2}} \phi^{q'-1}(x) dx \\ &\leq 2c_{\alpha} \int_{\Omega} \mathcal{D}_t u(x) \mathcal{D}_{2s-t}\phi(x) \phi^{q'-1}(x) dx \leq \mu_1 \int_{\Omega} (\mathcal{D}_t u(x))^q \phi^{q'}(x) dx + \tilde{c}_{\alpha, \mu_1} \int_{\Omega} (\mathcal{D}_{2s-t}\phi(x))^{q'} dx. \end{aligned}$$

Hence, combining the above estimates, we deduce that

$$\tilde{c}_{\alpha, \mu_1} \int_{\Omega} (\mathcal{D}_{2s-t}\phi(x))^{q'} dx \geq \lambda \int_{\Omega} f(x)\phi^{q'}(x)dx.$$

Thus necessarily

$$\lambda \leq \inf_{\{\phi \in C^{\infty}(\Omega), \phi \geq 0\}} \frac{\tilde{c}_{\alpha, \mu_1} \int_{\Omega} (\mathcal{D}_{2s-t}\phi(x))^{q'} dx}{\int_{\Omega} f(x)\phi^{q'}(x)dx} := \lambda^{**}.$$

□

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