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Asymptotically Linear Problems and Antimaximum Principle for the Square Root of the Laplacian

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David Arcoya

Departamento de Análisis Matemático, Universidad de Granada
Facultad de Ciencias, Campus Fuentenueva S/N 18071, Granada, España
e-mail: darcoya@ugr.es

Eduardo Colorado, Tommaso Leonori *

Departamento de Matemáticas, Universidad Carlos III de Madrid
Avda. Universidad 30, 28911 Leganés (Madrid), España.
e-mail: eduardo.colorado@uc3m.es, tommaso.leonori@uc3m.es

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Abstract

This work deals with bifurcation of positive solutions for some asymptotically linear problems, involving the square root of the Laplacian $(-\Delta)^{1/2}$. A simplified model problem is the following:

$$\begin{cases} (-\Delta)^{1/2}u = \lambda m(x)u + g(u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

with $\Omega \subset \mathbb{R}^N$ a smooth bounded domain, $N \geq 2$, $\lambda > 0$, $m \in L^\infty(\Omega)$, $m^+ \not\equiv 0$ and g is a continuous function which is super-linear at 0 and sub-linear at infinity. As a consequence of our bifurcation theory approach we prove some existence and multiplicity results. Finally, we also show an anti-maximum principle in the corresponding functional setting.

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1 Introduction

This work is motivated by the well known paper [3], where Ambrosetti and Hess used bifurcation theory to study the existence of positive solutions of Dirichlet boundary value problems for second order linear operators with asymptotically linear nonlinearities. The elegant and in depth approach in this pioneering paper has inspired several extensions of such a result to more general *"local"* differential operators, like, for instance the p-Laplacian (see [1, 2, 6]) and non homogeneous quasilinear operators like $-\mathrm{div}\,(A(x,u)\nabla u)$ with A an elliptic matrix (see [5]).

Our aim is to show that this nowadays classic approach can also be useful to handle nonlocal differential operators like the fractional Laplacian. These nonlocal diffusive operators have recently attracted the attention of the mathematical community (see among the others [8, 9, 10, 11, 12]) motivated by several physical phenomena like flames propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics, also in probability, American option in finance and also in α-stable Lévy processes (see for instance [4, 7, 13, 19]).

This paper deals with a nonlinear problem involving the square root of the Laplacian $(-\Delta)^{1/2}$ and an asymptotically linear reaction term. Clearly, the difficulties in order to face a problem of this kind are twofold. The first one is due to the presence on the right hand side of a nonlinear term. On the other hand we have to handle a nonlocal operator and we need to adapt the classical techniques of nonlinear analysis to this functional setting.

Specifically, we consider the nonlinear Dirichlet boundary value problem

$$\begin{cases} (-\Delta)^{1/2}u = f(\lambda, x, u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 2$, $f(\lambda, x, u)$ is a Carathéodory function in $[0,\infty)\times\Omega\times\mathbb{R}$ such that $f(\lambda, x, 0) \geq 0$ for all $\lambda \geq 0$, $x \in \Omega$ and

$$f(\lambda, x, s) = \lambda\, m(x)s + g(\lambda, x, s)$$

with $m \in L^\infty(\Omega)$, $m^+ \not\equiv 0$ and

$$|g(\lambda, x, s)| \leq \alpha(\lambda)\gamma(x)\mathcal{G}(s),$$

for some $\gamma \in L^\infty(\Omega)$ and $\alpha, \mathcal{G} \in C([0,\infty);\mathbb{R})$ satifying that either (f is asymptotically linear at zero)

$$\lim_{s\to 0} \frac{\mathcal{G}(s)}{s} = 0\,,$$

or (f is asymptotically linear at infinity)

$$\lim_{s\to +\infty} \frac{\mathcal{G}(s)}{s} = 0\,.$$

Denoting by $\lambda_{1,m}$ the first positive eigenvalue of the square root of the Laplacian $(-\Delta)^{1/2}$ with zero Dirichlet boundary conditions (see Proposition 2.2 below), we show (Theorem 3.1) that if f is asymptotically linear at zero, there exists a global branch of nonnegative (and nontrivial) solutions to (1.1) that emanates from $(\lambda_{1,m}, 0)$. Similarly (see Theorem 3.2) we deduce the existence of a branch of nonnegative solutions emanating from infinity at $\lambda_{1,m}$, provided that f has an asymptotically linear behavior at infinity.

By combining these two results together with the study of the side of the bifurcation at the respective bifurcation points (see Theorems 4.1 and 4.2 below), we deduce several results about the existence and multiplicity of positive solutions of (1.1) in Corollaries 4.1, 4.2 and 4.3. Furthermore, we also apply the above bifurcation results to deduce an anti-maximum principle for the fractional Laplacian operator $(-\Delta)^{1/2}$ (see Theorem 5.1).

The paper contains 4 more sections. In Section 2, in order to present the results in a self-contained way, we collect some basic tools about the square root of the Laplacian operator $(-\Delta)^{1/2}$ including its spectral definition. Section 3 deals with the results on bifurcation from zero or infinity. In Section 4, we study the side on which the bifurcation occurs, and as a consequence, we prove some existence and multiplicity results. Finally, the anti-maximum principle is studied in Section 5.

2 Preliminaries

In this preliminary section we start with the definition of the square root of the Laplacian operator $(-\Delta)^{1/2}$. We remember that one way to define the fractional powers of a positive operator is through its spectral decomposition, taking the powers of the associated eigenvalues. Indeed, let $\{\mu_k\}_{k\in\mathbb{N}}$ be the nondecreasing sequence of (positive) eigenvalues with associated eigenfunctions ϕ_k of the Laplace operator in a bounded domain Ω with zero Dirichlet boundary data, i.e.,

$$\begin{cases} -\Delta\phi_k = \mu_k\phi_k & \text{in } \Omega, \\ \phi_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We assume that the eigenfunctions ϕ_k are normalized by $\|\phi_k\|_{L^2(\Omega)} = 1$. Let us consider the space of functions

$$H_0^{1/2}(\Omega) = \left\{ u = \sum_{k=1}^{\infty} a_k\phi_k \ : \ \left(\sum_{k=1}^{\infty} a_k^2\mu_k^{1/2}\right)^{1/2} < \infty \right\},$$

with the norm

$$\|u\|_{H_0^{1/2}(\Omega)} = \left(\sum_{k=1}^{\infty} a_k^2\mu_k^{1/2}\right)^{1/2}.$$

The square root of the Laplacian $(-\Delta)^{1/2}$ is defined by

$$(-\Delta)^{1/2}u = \sum_{k=1}^{\infty} a_k\mu_k^{1/2}\phi_k\,,$$

for every $u = \sum_{k=1}^{\infty} a_k\phi_k \in H_0^{1/2}(\Omega)$. Clearly, $(-\Delta)^{1/2}$ maps $H_0^{1/2}(\Omega)$ into its dual space $H^{-1/2}(\Omega)$, being $\{\mu_k^{1/2}, \phi_k\}_{k\in\mathbb{N}}$ the eigenvalues and eigenfunctions of $(-\Delta)^{1/2}$, with zero Dirichlet boundary conditions. Note that $\|u\|_{H_0^{1/2}(\Omega)} = \|(-\Delta)^{1/4}u\|_{L^2(\Omega)}$ where, in general, the fractional powers $(-\Delta)^s$, $0 < s < 1$, can be defined in a similar way as the square root of $(-\Delta)^{1/2}$, following the previous process by taking the powers of the eigenvalues (notice that the spectrum in this case is defined by $\{\mu_k^s, \phi_k\}_{k=1}^{\infty}$, see [8]). From now on we will denote $\lambda_k = \mu_k^{1/2}$, $k \in \mathbb{N}$.

Actually, there exists another way of computing the square root of the Laplacian in $\mathbb{R}^N$. It proceeds through the so-called Dirichlet to Neumann operator by Stein [18], which is based on the harmonic extension in one more variable in $\mathbb{R}_+^{N+1}$. See also the paper by Caffarelli and Silvestre [11] for a generalized procedure when one consider other fractional powers of the Laplacian. In our case, for bounded domains, that extension can be seen in [8, 9, 10, 12].

Associated to the bounded domain Ω, let us consider the cylinder $C_\Omega = \Omega \times (0,\infty) \subset \mathbb{R}^{N+1}_+$. The points in C_Ω are denoted by $(x,y) \in \Omega \times (0,\infty)$. The lateral boundary of the cylinder will be denoted by $\partial_L C_\Omega = \partial\Omega \times (0,\infty)$.

Now, for a function $u \in H_0^{1/2}(\Omega)$, we define its *harmonic extension* $w = \mathrm{E}(u)$ to the cylinder C_Ω as the unique solution to the problem

$$\begin{cases} -\Delta w = 0 & \text{in } C_\Omega, \\ w = 0 & \text{on } \partial_L C_\Omega, \\ w = u & \text{on } \Omega \times \{y = 0\}. \end{cases} \tag{2.2}$$

The extension function w belongs to the space $X_0(\Omega)$ defined as the completion of $C_0^\infty(\Omega \times [0,\infty))$ endowed with the norm

$$\|w\|_{X_0(C_\Omega)} = \left(\int_{C_\Omega} |\nabla w|^2 \right)^{1/2}.$$

Hence the extension operator E is an isometry between $H_0^{1/2}(\Omega)$ and $X_0(C_\Omega)$, that is,

$$\| \mathrm{E}(u)\|_{X_0(C_\Omega)} = \|u\|_{H_0^{1/2}(\Omega)}, \quad \forall\, u \in H_0^{1/2}(\Omega). \tag{2.3}$$

Moreover, for any function $w \in X_0(C_\Omega)$, we have the following inequality for the trace $w(\cdot,0)$

$$\|w(\cdot,0)\|_{H_0^{1/2}(\Omega)} \le \|w\|_{X_0(C_\Omega)}.$$

The relevance of the extension function w is that it is related to the fractional Laplacian of the original function u through the formula

$$-\lim_{y\to 0^+} \frac{\partial w}{\partial y}(x,y) = -\frac{\partial w}{\partial y}(x,0) = (-\Delta)^{1/2} u(x),$$

see for example [8, 10, 11, 12, 18] for more details.

If f belongs to the dual space $H^{-1/2}(\Omega)$ of $H_0^{1/2}(\Omega)$, we consider the Dirichlet problem

$$\begin{cases} (-\Delta)^{1/2} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Since the definition of the fractional Laplacian allows to integrate by parts in the proper spaces, a natural definition of energy solution to problem (2.4) is the following.

Definition 2.1 *We say that $u \in H_0^{1/2}(\Omega)$ is an energy solution of (2.4) if the identity*

$$\int_\Omega (-\Delta)^{1/4} u (-\Delta)^{1/4} \varphi = \int_\Omega f\varphi$$

holds for every test function $\varphi \in H_0^{1/2}(\Omega)$.

The existence and uniqueness of an energy solution for every $f \in H^{-1/2}(\Omega)$ has been proved in [10]. This means that the inverse operator $K = (-\Delta)^{-1/2}$ is well defined from $H^{-1/2}(\Omega)$ into $H_0^{1/2}(\Omega)$.

On the other hand, by using the Dirichlet to Neumann operator, we can also characterized the solutions of (2.4) through the problem

$$\begin{cases} -\Delta w = 0 & \text{in } C_\Omega, \\ w = 0 & \text{on } \partial_L C_\Omega, \\ -\dfrac{\partial w}{\partial y}(x,0) = f & \text{on } \Omega \times \{y = 0\}. \end{cases}$$

Indeed, if $w \in X_0(C_\Omega)$ is an energy solution of this problem, i.e. if

$$\int_{C_\Omega} \langle \nabla w, \nabla \varphi \rangle \, dxdy = \int_\Omega f \varphi \, dx, \qquad \forall \, \varphi \in X_0(C_\Omega),$$

then the trace function $u = w(\cdot, 0)$ belongs to the space $H_0^{1/2}(\Omega)$ and it is an energy solution of problem (2.4). The converse is also true.

Another tool which is very useful in what follows are both the fractional Sobolev inequality, namely for every function $v \in H_0^{1/2}(\Omega)$ we have

$$\int_\Omega |(-\Delta)^{1/4} v|^2 \, dx \geq C \left(\int_\Omega |v|^r \, dx \right)^{2/r} \tag{2.5}$$

for any $1 \leq r \leq 2^*$, where $2^* = \frac{2N}{N-1}$ denotes the critical Sobolev exponent. We also have the following trace inequality

$$\int_{C_\Omega} |\nabla z(x,y)|^2 \, dxdy \geq C \left(\int_\Omega |z(x,0)|^r \, dx \right)^{2/r}, \tag{2.6}$$

for any $1 \leq r \leq 2^*$, and any $z \in X_0(C_\Omega)$, for a universal constant $C = C(r, N, \Omega) > 0$. We remember that when $r = 2^*$, the best constant $S(N)$ in (2.5)-(2.6) is not achieved in any bounded domain. For such results we refer to [16].

Note that by the Sobolev inequality (2.5) we have the continuous embedding $H_0^{1/2}(\Omega) \hookrightarrow L^r(\Omega)$ for any $1 \leq r \leq 2^*$, which clearly is compact for $r < 2^*$. Using strongly the harmonic extension previously defined, Cabré and Tan (see Proposition 3.1 in [10]) proved the following regularity result which will be useful in the sequel.

We denote by $C_0(\overline{\Omega})$ (resp. $C_0^\beta(\overline{\Omega})$, $0 < \beta < 1$) the space of continuous (resp. Hölder continuous) functions on $\overline{\Omega}$ that vanish on the boundary.

Proposition 2.1 ([10]) *Let u be the solution to problem (2.4) and $0 < \beta < 1$, then there exists a constant $c > 0$ such that:*

i) If $f \in L^\infty(\Omega)$ we have that $u \in C^\beta(\overline{\Omega})$, and $\|u\|_{C^\beta(\overline{\Omega})} \leq c\|f\|_{L^\infty(\overline{\Omega})}$.

ii) If $f \in C_0^\beta(\overline{\Omega})$ we have that $u \in C^{1,\beta}(\overline{\Omega})$, and $\|u\|_{C^{1,\beta}(\overline{\Omega})} \leq c\|f\|_{C^\beta(\overline{\Omega})}$.

Remark 2.1 For every $f \in H^{-1/2}(\Omega)$, let Kf be defined as the unique solution $u \in H_0^{1/2}(\Omega)$ of (2.4). In this way, K (the inverse of the square root of the Laplacian operator) maps $H^{-1/2}(\Omega)$ into $H_0^{1/2}(\Omega)$. We observe that, from Proposition 2.1, we deduce that K is continuous from continuity of $L^\infty(\Omega)$ into $C_0^\beta(\overline{\Omega})$, which together to the compact imbedding of $C_0^\beta(\overline{\Omega})$ into $C_0(\overline{\Omega})$ implies that K is a compact operator from $C_0(\overline{\Omega})$ into itself.

Now, we state a result that relies to be classical (see [14] for instance) in the case of linear second order elliptic operator that involves eigenvalues and eigenfunctions with weights.

Proposition 2.2 *Assume that $m(x) \in L^r(\Omega)$, for some $r > N$ and $m^+ \not\equiv 0$ a.e. in Ω. Then the weighted eigenvalue problem*

$$\begin{cases} (-\Delta)^{1/2}\varphi = \lambda\, m(x)\,\varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.7}$$

has a first positive and simple eigenvalue $\lambda_{1,m}$, and an associated eigenfunction $\varphi = \varphi_{1,m} \in H_0^{1/2}(\Omega)$, which can be taken, without loss of generality, positive.

Proof. For every function $u \in H_0^{1/2}(\Omega)$ we define $v = Tu \in H_0^{1/2}(\Omega)$, as the solution in $H_0^{1/2}(\Omega)$ of $(-\Delta)^{1/2}v = mu$. Clearly $T : H_0^{1/2}(\Omega) \longrightarrow H_0^{1/2}(\Omega)$ is a linear, symmetric and bounded operator. Furthermore, T is a compact operator, indeed, if $\{u_n\}_{n=1}^\infty \subset H_0^{1/2}(\Omega)$ is a bounded sequence, then there exists a subsequence $u_n \rightharpoonup u_0$ weakly in $H_0^{1/2}(\Omega)$, and moreover, $u_n \to u_0$ strongly in $L^p(\Omega)$ for $p < 2^*$. Taking into account also that $r > N$, there exists $1 < s < 2^*$ with $\frac{1}{s} = 1 - \frac{1}{r} - \frac{1}{2^*}$ such that

$$\|Tu_n - Tu_0\|^2 \le \|m\|_{L^r(\Omega)}\|u_n - u_0\|_{L^s(\Omega)}\|Tu_n - Tu_0\|_{L^{2^*}(\Omega)}.$$

Using the Sobolev inequality (2.5) in the previous expression, we obtain that $Tu_n \to Tu_0$ in $H_0^{1/2}(\Omega)$.

Observing that every eigenvalue λ of (2.7) is different from zero, problem (2.7) can be rewritten as $Tu = \frac{1}{\lambda}u$ and the spectral theory of compact symmetric operators can be applied to characterize the eigenvalues of (2.7) (see Section 1.1 in [14]). In particular, $\frac{1}{\lambda_{1,m}} := \sup\{(Tv, v)_{H_0^{1/2}(\Omega)} : \|v\|_{H_0^{1/2}(\Omega)} = 1\}$ is the biggest eigenvalue of T, i.e.,

$$\lambda_{1,m} = \inf_{v \in H_0^{1/2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |(-\Delta)^{1/4}v|^2\,dx}{\displaystyle\int_\Omega m(x)v^2\,dx} \tag{2.8}$$

corresponds to the first positive eigenvalue of (2.7). We observe that this eigenvalue also can be characterized (using the harmonic extension and (2.3)) as

$$\lambda_{1,m} = \inf_{v \in H_0^{1/2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_{C_\Omega} |\nabla\, \mathrm{E}(v)|^2\,dxdy}{\displaystyle\int_\Omega m(x)v^2\,dx} = \inf_{w \in X_0(C_\Omega)\setminus\{0\}} \frac{\displaystyle\int_{C_\Omega} |\nabla w|^2\,dxdy}{\displaystyle\int_\Omega m(x)w^2(x,0)\,dx}, \tag{2.9}$$

where $w(x, 0)$ denotes the trace of w on $\Omega \times \{0\}$.

We prove now that every eigenfunction φ corresponding to $\lambda_{1,m}$ does not change sign in Ω. More precisely, assume that $\varphi \not\equiv 0$ is an eigenfunction corresponding to the first eigenvalue $\lambda_{1,m}$, i.e., φ is a minimum of the minimization problem (2.8). Clearly, if $w = E(\varphi) \in X_0(C_\Omega)$, the Rayleigh quotient in (2.9) is the same for w and $|w|$, i.e.,

$$\frac{\displaystyle\int_{C_\Omega} |\nabla w|^2\, dxdy}{\displaystyle\int_\Omega m(x) w^2(x,0)\, dx} = \frac{\displaystyle\int_{C_\Omega} |\nabla |w||^2\, dxdy}{\displaystyle\int_\Omega m(x) |w|^2(x,0)\, dx}.$$

As a consequence, $|w|$ is also a minimizer for (2.9) and $|\varphi| = |w(x,0)|$ is an eigenfunction associated to $\lambda_{1,m}$ (since it is a minimizer for (2.8)).

By the strong maximum principle for the square root of the Laplacian $(-\Delta)^{1/2}$, we deduce that $|\varphi| > 0$ in Ω. As a consequence, either $\varphi > 0$ or $\varphi < 0$ in Ω.

The proof that $\lambda_{1,m}$ has one geometric multiplicity follows like in the second step of the proof of Theorem 1.13 in [14]. As a direct application, the algebraic multiplicity of $\lambda_{1,m}$ has to be also one. □

Remark 2.2 Note that in the above proof we have used the maximum principle for the square root of the Laplacian $(-\Delta)^{1/2}$ in Ω. This result (see [10] for the detailed proof) can be shown through the classical maximum principle applied to the Stein harmonic extension in the cylinder C_Ω.

3 Asymptotically linear problems

In this section we consider the existence of positive solutions (i.e. nonnegative and nonzero solutions) of the following nonlinear problem:

$$(P_\lambda) \qquad \begin{cases} (-\Delta)^{1/2} u = f(\lambda, x, u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where

$$\begin{aligned} f : [0,+\infty) \times \Omega \times [0,+\infty) \to \mathbb{R} \text{ is a Caratheodory function and} \\ f(\lambda, x, 0) \geq 0\,, \quad \forall (\lambda, x) \in \mathbb{R}^+ \times \Omega\,. \end{aligned} \tag{3.1}$$

We will make suitable assumptions over f later.

Remark 3.1 In the sequel we always extend the function $f(\lambda, x, s)$ (which is only defined for positive $s \geq 0$) to all $[0,+\infty) \times \Omega \times \mathbb{R}$ by

$$f(\lambda, x, s) = \begin{cases} f(\lambda, x, s) & \text{if } s \geq 0\,, \\ f(\lambda, x, 0) & \text{if } s < 0\,. \end{cases}$$

Since $f(\lambda, x, 0) \geq 0$, by the maximum principle (see [9]), every nonzero solution of (P_λ) with this extended nonlinearity f relies to be positive.

By using some arguments dealing with the topological degree of Leray-Schauder (see [1, 15, 17] for the definition and basic properties), we prove the first result concerning the bifurcation from zero. We use the following notation for the set of solutions of (P_λ):

$$\Sigma := \overline{\{(\lambda, u) \in [0, +\infty) \times C(\overline{\Omega}) : u \text{ solves } (P_\lambda),\ u \not\equiv 0\}}.$$

Theorem 3.1 *Assume that (3.1) holds true and that*

$$f(\lambda, x, s) = \lambda\, m_0(x)s + g_0(\lambda, x, s)$$

where $m_0 \in L^\infty(\Omega)$, $m_0^+ \not\equiv 0$, and $g_0 : [0, +\infty) \times \Omega \times [0, +\infty) \to \mathbb{R}$ is a Caratheodory function that satisfies

$$g_0(0, x, s) \equiv 0 \quad \textit{and} \quad |g_0(\lambda, x, s)| \leq \alpha(\lambda)\gamma(x)\mathcal{G}_0(s)\,, \tag{3.2}$$

for some $\gamma(x) \in L^\infty(\Omega)$ and α, $\mathcal{G}_0 \in C(\mathbb{R}^+; \mathbb{R})$ such that:

$$\lim_{s\to 0^+} \frac{\mathcal{G}_0(s)}{s} = 0\,. \tag{3.3}$$

Then there exists an unbounded continuum $\Sigma_0 \subset \Sigma$ that emanates from $(\lambda_{1,m_0}, 0)$.

Proof. We divide the proof into three steps.

Step 1. *For any compact subset $\Lambda \subset [0, +\infty)$ such that $\lambda_{1,m_0} \notin \Lambda$, there exists $\varepsilon > 0$ such that problem (P_λ) has no nonzero solution u with $\|u\|_\infty \leq \varepsilon$ and $\lambda \in \Lambda$.*

By contradiction, suppose that there exist $u_n \in C(\overline{\Omega})$ and $\lambda_n \in \Lambda$ which solve problem

$$\begin{cases} (-\Delta)^{1/2}u_n = \lambda_n\, m_0(x)u_n + g_0(\lambda_n, x, u_n) & \text{in } \Omega\,, \\ u_n = 0 & \text{on } \partial\Omega\,, \end{cases}$$

with $u_n \not\equiv 0$, $\|u_n\|_\infty \to 0$ and $\lambda_n \to \lambda_0 \in \Lambda$. Thus, the functions $z_n = \frac{u_n}{\|u_n\|_\infty}$ satisfy

$$\begin{cases} (-\Delta)^{1/2}z_n = \lambda_n\, m_0(x)z_n + \dfrac{g_0(\lambda_n, x, u_n)}{\|u_n\|_\infty} & \text{in } \Omega\,, \\ z_n = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3.4}$$

By properties (3.2) and (3.3), we deduce that $\frac{g_0(\lambda_n,x,u_n)}{\|u_n\|_\infty} \to 0$ in $C(\overline{\Omega})$. Hence, using z_n as a test function in (3.4) and by (2.5) one easily gets that there exists a constant $C > 0$ such that

$$\int_\Omega |(-\Delta)^{1/4}z_n|^2 \leq C.$$

As a consequence, up to a subsequence (if necessary), we can assume that $z_n \rightharpoonup z_0$ weakly in $H_0^{1/2}(\Omega)$ with z_0 satisfying

$$\begin{cases} (-\Delta)^{1/2}z_0 = \lambda_0 m_0(x)z_0 & \text{in } \Omega\,, \\ z_0 = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3.5}$$

Using that $H_0^{1/2}(\Omega)$ is compactly embedded in $L^p(\Omega)$ for any $1 < p < \frac{2N}{N-1}$, (see Section 2), and that $\|z_n\|_\infty = 1$, we also deduce that $\|z_0\|_\infty = 1$, and $z_0 \geq 0$ (see Remark 2.1). Hence, by Proposition 2.2, necessarily $\lambda_0 = \lambda_{1,m_0} \in \Lambda$, a contradiction.

Step 2. *For any* $\lambda > \lambda_{1,m_0}$, *there exists* $\varepsilon > 0$ *such that problem*

$$\begin{cases} (-\Delta)^{1/2}u = \lambda\, m_0(x)u + g_0(\lambda, x, u) + t m_0(x)\varphi_{1,m_0} & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.6}$$

does not posses a solution with $0 < \|u\|_\infty \le \varepsilon$ *and* $t \ge 0$.

By contradiction, we suppose that there exist sequences $u_n \not\equiv 0$ and $t_n \ge 0$ such that

$$\begin{cases} (-\Delta)^{1/2}u_n = \lambda\, m_0(x)u_n + g_0(\lambda, x, u_n) + t_n m_0(x)\varphi_{1,m_0} & \text{in } \Omega\,, \\ u_n = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.7}$$

with $\|u_n\|_\infty \to 0$. Using φ_{1,m_0} as test function in (3.7), we deduce that $z_n = \frac{u_n}{\|u_n\|_\infty}$ satisfies

$$(\lambda_{1,m_0} - \lambda) \int_\Omega m_0(x) z_n \varphi_{1,m_0} - \int_\Omega \frac{g_0(\lambda, x, u_n)}{\|u_n\|_\infty}\varphi_{1,m_0} = \frac{t_n}{\|u_n\|_\infty} \int_\Omega m_0(x)\varphi_{1,m_0}^2\,.$$

Notice that the left hand side in the previous identity is bounded (with respect to n), so that necessarily $t_n/\|u_n\|_\infty$ is bounded. Passing to a subsequence, if necessary, we deduce that $t_n/\|u_n\|_\infty \to t_0 \ge 0$. In addition, by similar arguments to the ones used in the previous step, we can also assume that $z_n = \frac{u_n}{\|u_n\|_\infty} \rightharpoonup z_0$ weakly in $H_0^{1/2}(\Omega)$ with z_0 satisfying

$$\begin{cases} (-\Delta)^{1/2}z_0 = \lambda m_0(x)z_0 + t_0 m_0(x)\varphi_{1,m_0} & \text{in } \Omega\,, \\ z_0 = 0 & \text{on } \partial\Omega\,, \end{cases}$$

and $\|z_0\|_\infty = 1$ as a consequence of the compactness of the operator K (see Remark 2.1). Since this problem is not solvable for $\lambda > \lambda_{1,m_0}$ and $t_0 \ge 0$, we get a contradiction concluding the proof of the second step.

Step 3. *Conclusion.*

In this step we apply the results of Step 1 and Step 2 in order to prove the existence of a continuum of solutions via a topological degree argument.

We first define $\Phi \in C(\mathbb{R} \times C(\overline{\Omega}); C(\overline{\Omega}))$ by

$$\Phi(\lambda, u) = u - \Psi_\lambda(u)\,,$$

where $\Psi_\lambda(u) = K(\lambda\, m_0(x)u + g_0(\lambda, x, u))$ is a compact operator, thanks to Proposition 2.1 (see also Remark 2.1).

In Step 1 we have proved that for any $\lambda < \lambda_{1,m_0}$ there exists a small ball B_ε such that if $\overline{\lambda} \in [0, \lambda]$, then the only zero u of $\Phi(\overline{\lambda}, \cdot)$ with norm smaller than ε is $u = 0$ since, by (3.2), $g(0, x, u) \equiv 0$ and $\Phi(0, \cdot)$ is the identity map. Hence, by the homotopy invariance property of the topological degree, we obtain

$$\deg(\Phi(\lambda, \cdot), B_\varepsilon, 0) = \deg(\Phi(0, \cdot), B_\varepsilon, 0) = 1\,.$$

On the other hand, in Step 2 we have proved that there exists $\varepsilon > 0$ such that for any $t \ge 0$, problem (3.6) does not posses nonzero solutions with norm smaller than ε if $\lambda > \lambda_{1,m_0}$. If we consider

$$\Phi_t(\lambda, u) = u - K(\lambda\, m_0(x)u + g_0(\lambda, x, u) + t m_0(x)\varphi_{1,m_0})\,,$$

again by the homotopy invariance property of the topological degree, this implies that

$$\deg(\Phi_t(\lambda,\cdot), B_\varepsilon, 0) = 0\,, \quad \forall \lambda > \lambda_{1,m_0}\,, \ \ \forall t \geq 0,$$

(since $\Phi_t(\lambda,\cdot)$ has no zeroes for $t > 0$). In particular, by choosing $t = 0$, we deduce that the degree $\deg(\Phi(\lambda,\cdot), B_\varepsilon, 0) = 0$, if $\lambda > \lambda_{1,m_0}$.

Hence we conclude, by the Global Bifurcation Theorem by Rabinowitz (see Theorem 3.7 in [17]), that there exists an unbounded continuum of solutions that emanates from $(\lambda_{1,m_0}, 0)$. □

Remark 3.2 Observe that by the above proof it is clear that λ_{1,m_0} is the unique positive bifurcation point from zero for (P_λ).

Remark 3.3 We want to stress an important tool that will be useful later. Notice that (3.4) together with (3.3) yields to $z_n = \frac{u_n}{\|u_n\|_\infty} \rightharpoonup z_0$ weakly in $H_0^{1/2}(\Omega)$. Furthermore, recalling that K is compact between $L^\infty(\Omega)$ and $C(\overline{\Omega})$ (see Remark 2.1) and observing that

$$z_n = K\Big(\lambda_n\, m_0(x) z_n + \frac{g_0(\lambda, x, u_n)}{\|u_n\|_\infty}\Big),$$

with $\lambda_0\, m_0(x) z_n + \frac{g_0(\lambda,x,u_n)}{\|u_n\|_\infty}$ uniformly bounded, we deduce that z_n uniformly converges to the solution z_0 of (3.5) and since $z_n \geq 0$ it follows that $z_0 \equiv \varphi_{1,m_0}$. This implies that $\|z_n - \varphi_{1,m_0}\|_\infty \to 0$.

Let us prove now the existence of a continuum of solutions that emanates from infinity.

Theorem 3.2 *Assume (3.1) holds true and that*

$$f(\lambda, x, s) = \lambda\, m_\infty(x) s + g_\infty(\lambda, x, s)$$

where m_∞ belongs to $L^\infty(\Omega)$, $m_\infty^+ \not\equiv 0$, and $g_\infty : [0,+\infty) \times \Omega \times [0,+\infty) \to \mathbb{R}$ is a Caratheodory function satisfying

$$g_\infty(0, x, s) \equiv 0 \quad \textit{and} \quad |g_\infty(\lambda, x, s)| \leq \alpha(\lambda)\gamma(x)\mathcal{G}_\infty(s) \tag{3.8}$$

for some $\gamma(x) \in L^\infty(\Omega)$ and $\alpha, \mathcal{G}_\infty \in C(\mathbb{R}^+;\mathbb{R})$ such that:

$$\lim_{s\to+\infty} \frac{\mathcal{G}_\infty(s)}{s} = 0\,. \tag{3.9}$$

Then there exists an unbounded set $\Sigma_\infty \subset \Sigma$ such that

$$\widetilde{\Sigma}_\infty = \Big\{(\lambda, u) \in \Sigma \,:\, (\lambda, u/\|u\|_\infty^2) \in \Sigma_\infty\Big\} \cup (\lambda_{1,m_\infty}, 0)$$

is an unbounded continuum.

Remark 3.4 Notice that the fact that $(\lambda_{1,m_\infty}, 0)$ belongs to the continuum $\widetilde{\Sigma}_\infty$ means that Σ_∞ emanates from $(\lambda_{1,m_\infty}, \infty)$.

Proof. We follow the same idea of the proof of Theorem 3.1.

Step 1. *For any compact subset* Λ *of* $[0,+\infty)$ *such that* $\lambda_{1,m_\infty} \notin \Lambda$*, there exists* $r > 0$ *such that the problem*

$$\begin{cases} (-\Delta)^{1/2}u = \lambda\, m_\infty(x)u + g_\infty(\lambda,x,u) + t m_\infty(x)\varphi_{1,m_\infty} & \text{in } \Omega\,,\\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

has no solution u with $\|u\|_\infty \geq r$ *and* $t \in [0,1]$.

By contradiction, we suppose that there exist sequences $\lambda_n \in \Lambda$, $t_n \in [0,1]$ and u_n solving the problem

$$\begin{cases} (-\Delta)^{1/2}u_n = \lambda_n\, m_\infty(x)u_n + g_\infty(\lambda,x,u_n) + t_n m_\infty(x)\varphi_{1,m_\infty} & \text{in } \Omega\,,\\ u_n = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.10}$$

such that $\lambda_n \to \lambda \neq \lambda_{1,m_\infty}$ and $\|u_n\|_\infty \to +\infty$. It is clear that $v_n = \frac{u_n}{\|u_n\|_\infty}$ is a solution of

$$\begin{cases} (-\Delta)^{1/2}v_n = \lambda_n\, m_\infty(x)v_n + \dfrac{g_\infty(\lambda,x,u_n)}{\|u_n\|_\infty} + \dfrac{t_n}{\|u_n\|_\infty} m_\infty(x)\varphi_{1,m_\infty} & \text{in } \Omega,\\ v_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.11}$$

with $v_n \geq 0$ and $\|v_n\|_\infty = 1$. Notice that by (3.8) and (3.9), we have

$$\lim_{n\to+\infty} \frac{g_\infty(\lambda_n,x,u_n)}{\|u_n\|_\infty} = 0,$$

uniformly in Ω and $t_n/\|u_n\|_\infty \to 0$ (since $\{t_n\}$ is bounded). Consequently, using v_n as a test function in (3.11) we deduce that the sequence $\{v_n\}$ is bounded in the energy space. Hence there exists $v_0 \in H_0^{1/2}(\Omega)$ such that $v_n \rightharpoonup v_0$ weakly in $H_0^{1/2}(\Omega)$, $0 \not\equiv v_0 \geq 0$ and it solves

$$\begin{cases} (-\Delta)^{1/2}v_0 = \lambda\, m_\infty(x)v_0 & \text{in } \Omega\,,\\ v_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

We get a contradiction since the above problem does not have any positive solution for $\lambda \neq \lambda_{1,m_\infty}$.

Step 2. *Conclusion.*

As in the previous theorem, we study the zeros of the function defined by $\Phi_0(\lambda,u) = u - K(\lambda\, m_\infty(x)u + g(\lambda,x,u))$ where K is given by Remark 2.1. As we have already noticed, it is an operator of the form $\Phi_0(u) = u - \Psi_\lambda(u)$ where Ψ_λ is a compact operator over $C(\overline{\Omega})$.

In order to apply the Global Bifurcation Theorem by Rabinowitz (see [17]), we perform the change of variable $z = \frac{u}{\|u\|_\infty^2}$. Thus we change the bifurcation from infinity for u to the bifurcation from 0 for z by defining the following function

$$\widetilde{\Phi}_t(\lambda,z) = \begin{cases} z - K\left(\lambda m_\infty(x)z + \|z\|_\infty^2\, g(\lambda,x,z/\|z\|_\infty^2) + \|z\|_\infty^2\, t\, m_\infty(x)\varphi_{1,m_\infty}\right) & \text{if } z \neq 0,\\ 0 & \text{if } z = 0. \end{cases}$$

A first consequence of Step 1 is that, if we fix $\lambda < \lambda_{1,m_\infty}$, then for $\overline{\lambda} \in [0,\lambda]$, the only zero of $\widetilde{\Phi}_0(\overline{\lambda},\cdot)$ with small norm is the trivial one. As a consequence of the homotopy property, for every $\lambda < \lambda_{1,m_\infty}$ we obtain

$$\deg(\widetilde{\Phi}_0(\lambda,\cdot), B_\varepsilon, 0) = \deg(\widetilde{\Phi}_0(0,\cdot), B_\varepsilon, 0) = 1\,, \quad \forall \varepsilon < \frac{1}{r}\,.$$

On the other hand, also by Step 1 and using one more time the homotopy invariance of the topological degree, we deduce that if $\lambda > \lambda_{1,m_\infty}$, then

$$\deg(\widetilde{\Phi}_0(\lambda,\cdot), B_\varepsilon, 0) = \deg(\widetilde{\Phi}_1(\lambda,\cdot), B_\varepsilon, 0) = 0\,, \quad \forall \varepsilon < \frac{1}{r}\,, \quad \forall \lambda > \lambda_{1,m_\infty}\,,$$

(where the last degree vanishes since $\widetilde{\Phi}_1(\lambda,\cdot)$ does not have any zero).

Hence, fixed a ball of radius small enough, we have a change of degree. Once again, by the Global Bifurcation Theorem by Rabinowitz in [17], we deduce the existence of an unbounded continuum of zeroes for $\widetilde{\Phi}_0(\lambda, z)$ that emanates from $(\lambda_{1,m_\infty}, 0)$, i.e. solutions of $\Phi_0(\lambda, u) = 0$ emanating from infinity at λ_{1,m_∞}. □

Remarks 3.1 1. As for the case of bifurcation from 0 (see Remark 3.2) we observe that from the proof of the above result we deduce that λ_{1,m_∞} is the unique positive bifurcation point from infinity for problem (P_λ).

2. We point out that if u_n is a sequence of solutions of (3.10), then we have:

$$\lim_{n\to\infty} \left\| \frac{u_n}{\|u_n\|_\infty} - \varphi_{1,m_\infty} \right\|_\infty = 0\,.$$

Indeed, following the proof of the first step in Theorem 3.2, if we define $v_n = \frac{u_n}{\|u_n\|_\infty}$, since $\|v_n\|_\infty = 1$ and $v_n \geq 0$, using the compactness of K in $C(\overline{\Omega})$ (see Remark 2.1), we conclude $v_n \to \varphi_{1,m_\infty}$ uniformly. In particular, this implies that $u_n \to \infty$ locally uniformly.

4 Sub and super criticality of the bifurcation

The study of the side in which a continuum emanates from a bifurcation point is very relevant to deduce existence and multiplicity of solutions of problem (P_λ). First, we point out some definitions.

Definition 4.1 *Let λ_0 be a bifurcation point from zero. We say that the bifurcation at λ_0 is subcritical (supercritical, respectively) if there exists a neighborhood U of $(\lambda_0, 0)$ such that for any nontrivial solution $(\lambda, u) \in U$ of (P_λ) we have $\lambda < \lambda_0$ ($\lambda > \lambda_0$, respectively).*

Analogously, if λ_∞ is a bifurcation point from infinity, we say that the bifurcation at λ_∞ is subcritical (supercritical, respectively) if there exists a neighborhood U of (λ_∞, ∞) such that for any solution $(\lambda, u) \in U$ of (P_λ) we have $\lambda < \lambda_\infty$ ($\lambda > \lambda_\infty$, respectively).

The first result we prove deals with sub and supercritical bifurcation from infinity. We recall that under the hypotheses of Theorem 3.2 we know that λ_{1,m_∞} is the (unique) positive bifurcation point from infinity

Theorem 4.1 *Assume that the hypotheses of Theorem 3.2 hold true. Suppose moreover that*

$$\begin{aligned} &\exists A_1(x) \in L^1(\Omega) \quad \text{such that } \forall \varepsilon > 0\, \exists s_0,\, \delta > 0 \quad \text{satisfying} \\ &g_\infty(\lambda, x, s) s^\eta \geq A_1(x) + \varepsilon\,, \qquad \forall s \geq s_0\,, \quad \forall \lambda \in (\lambda_{1,m_\infty} - \delta, \lambda_{1,m_\infty} + \delta)\,, \end{aligned} \tag{4.1}$$

and

$$\int_\Omega \underline{A}(x)\varphi_{1,m_\infty}^{1-\eta} > 0 \quad \textit{where} \quad \underline{A}(x) := \liminf_{(\lambda,s)\to(\lambda_{1,m_\infty},\infty)} g_\infty(\lambda, x, s)s^\eta\,, \tag{4.2}$$

for some $\eta > -1$ *(according with hypothesis (3.9)). Then the bifurcation from infinity at* λ_{1,m_∞} *is subcritical.*

On the other hand, if we suppose that

$$\begin{array}{c} \exists A_2(x) \in L^1(\Omega) \quad \textit{such that}\ \forall\varepsilon > 0\ \exists s_0,\ \delta > 0 \quad \textit{satisfying} \\ g_\infty(\lambda, x, s)s^\eta \le A_2(x) - \varepsilon\,, \qquad \forall s \ge s_0\,,\ \forall\lambda \in (\lambda_{1,m_\infty} - \delta, \lambda_{1,m_\infty} + \delta)\,, \end{array} \tag{4.3}$$

and

$$\int_\Omega \overline{A}(x)\varphi_{1,m_\infty}^{1-\eta} < 0\,, \quad \textit{where} \quad \overline{A}(x) := \limsup_{(\lambda,s)\to(\lambda_{1,m_\infty},\infty)} g_\infty(\lambda, x, s)s^\eta\,, \tag{4.4}$$

for some $\eta > -1$ *(according with hypothesis (3.9) again). Then supercritical bifurcation at* λ_{1,m_∞} *occurs.*

Proof. Let $\{u_n\}$ be a sequence of positive solutions to (P_{λ_n}) with $\lambda_n \neq \lambda_{1,m_\infty}$, $\lambda_n \to \lambda_{1,m_\infty}$ and $\|u_n\|_\infty \to \infty$. Thus, using $\varphi_{1,m_\infty}/\|u_n\|_\infty$ as test function in (P_λ), we obtain that

$$(\lambda_{1,m_\infty} - \lambda_n)\int_\Omega m_\infty(x)\, v_n\varphi_{1,m_\infty} = \int_\Omega \frac{g_\infty(\lambda_n, x, u_n)}{\|u_n\|_\infty}\varphi_{1,m_\infty}\,,$$

where $v_n = u_n/\|u_n\|_\infty$. By Remark 3.1, v_n converges to φ_{1,m_∞} uniformly, and then for n large enough the integral in the left hand side above is strictly positive. Consequently

$$\mathrm{sign}\,(\lambda_{1,m_\infty} - \lambda_n) = \ \mathrm{sign}\ \int_\Omega \frac{g_\infty(\lambda_n, x, u_n)}{\|u_n\|_\infty}\varphi_{1,m_\infty}\,. \tag{4.5}$$

In order to study the sub or super criticality of the bifurcation, we need to know the sign of the integral above for n large. Again by Remark 3.1, we know that v_n uniformly converges to φ_{1,m_∞} and consequently u_n blows up locally uniformly (i.e. uniformly in any compact of Ω). Hence, thanks to hypotheses (4.1) and (4.2), we can apply Fatou lemma to deduce that

$$\liminf_{n\to\infty}\int_\Omega \frac{g_\infty(\lambda_n, x, u_n)(1+u_n)^\eta}{(1/\|u_n\|_\infty + v_n)^\eta}\varphi_{1,m_\infty} \ge \int_\Omega \underline{A}(x)\varphi_{1,m_\infty}^{1-\eta} > 0\,.$$

Therefore, for n large enough, we have

$$\mathrm{sign}\int_\Omega \frac{g_\infty(\lambda_n, x, u_n)}{\|u_n\|_\infty}\varphi_{1,m_\infty} = \mathrm{sign}\frac{1}{\|u_n\|_\infty^{1+\eta}}\int_\Omega \frac{g_\infty(\lambda_n, x, u_n)(1+u_n)^\eta}{(1/\|u_n\|_\infty + v_n)^\eta}\varphi_{1,m_\infty} > 0.$$

Thus, recalling (4.5) we deduce that we have subcritical bifurcation from infinity.

The supercritical case follows by repeating the same proof and exploiting hypothesis (4.3) and (4.4) instead of (4.1) and (4.2). □

Similarly, we have the following result for the sub or super criticality of the bifurcation from zero.

Theorem 4.2 *Assume the hypotheses of Theorem 3.1 hold. Suppose moreover that*

$$\begin{gathered} \exists\, \eta < -1\,,\ \exists\, b(x) \in L^1(\Omega)\ :\ \forall \varepsilon > 0\, \exists\, s_0 > 0 \text{ and } \delta > 0\,; \\ |g_0(\lambda, x, s)||s|^{\eta} \le b(x) + \varepsilon\,, \end{gathered} \tag{4.6}$$

for any $0 < s \le s_0$ *and any* $\lambda \in (\lambda_{1,m_0} - \delta, \lambda_{1,m_0} + \delta)$ *and that*

$$\int_\Omega \underline{B}(x)\varphi_{1,m_0}^{1-\eta} > 0 \quad \text{where} \quad \underline{B}(x) := \liminf_{(\lambda,s)\to(\lambda_{1,m_0},0)} |g_0(\lambda, x, s)||s|^{\eta-1} s\,. \tag{4.7}$$

Then we have subcritical bifurcation from zero at λ_{1,m_0}.

On the other hand, if we assume that (4.6) and

$$\int_\Omega \overline{B}(x)\varphi_{1,m_0}^{1-\eta} < 0 \quad \text{where} \quad \overline{B}(x) := \limsup_{(\lambda,s)\to(\lambda_{1,m_0},0)} |g_0(\lambda, x, s)||s|^{\eta-1} s\,, \tag{4.8}$$

hold true, then supercritical bifurcation at λ_{1,m_0} *occurs.*

Proof. We follow the same idea of the previous result. Let us consider a sequence of nonzero solutions of (P_{λ_n}) with λ_n converging to λ_{1,m_0} and $\|u_n\|_\infty$ converging to zero. Using $\frac{\varphi_{1,m_0}}{\|u_n\|_\infty}$ as a test function, we deduce that

$$(\lambda_{1,m_0} - \lambda_n)\int_\Omega m_0(x)v_n\varphi_{1,m_0} = \frac{1}{\|u_n\|_\infty}\int_\Omega g_0(\lambda_n, x, u_n)\varphi_{1,m_0}\,, \tag{4.9}$$

where $v_n = u_n/\|u_n\|_\infty$. Our goal is to study the sign of $(\lambda_{1,m_0} - \lambda_n)$ as n diverges. As we have already observed (see Remark 3.3), by hypothesis (3.3) we get $v_n \to \varphi_{1,m_0}$ uniformly, then there exists $n_0 > 0$ such that

$$\int_\Omega m_0(x)v_n\varphi_{1,m_0} > 0\,, \qquad \forall n \ge n_0\,.$$

On the other hand, let us define $\Omega_n = \{x \in \Omega\ :\ u_n(x) = 0\}$ and observe that, by (3.3), $g_0(\lambda, x, u_n) = 0$ in Ω_n, so that

$$\frac{1}{\|u_n\|_\infty}\int_\Omega g_0(\lambda_n, x, u_n)\varphi_{1,m_0} = \frac{1}{\|u_n\|_\infty^{\eta+1}}\int_{\Omega_n} \frac{g_0(\lambda_n, x, u_n)u_n|u_n|^{\eta-1}}{v_n|v_n|^{\eta-1}}\varphi_{1,m_0}\,.$$

By (4.6) and Fatou lemma we deduce, since once again $v_n \to \varphi_{1,m_0}$ uniformly, that

$$\liminf_{n\to\infty}\int_{\Omega_n} \frac{g_0(\lambda_n, x, u_n)u_n|u_n|^{\eta-1}}{v_n|v_n|^{\eta-1}}\varphi_{1,m_0} \ge \int_\Omega \underline{B}(x)\varphi_{1,m_0}^{1-\eta} > 0\,,$$

thanks to (4.7). Hence the sign of the right hand side in (4.9) is positive and consequently $\lambda_{1,m_0} - \lambda_n > 0$. In conclusion, we have subcritical bifurcation.

The supercritical case follows by repeating the same proof and exploiting hypothesis (4.8) instead of (4.7). □

Now we list some consequences of the above results. We observe that, in principle, the sets Σ_0 and Σ_∞ given respectively by Theorems 3.1 and 3.2 do not coincide. Next result provides a sufficient condition in order to have that $\Sigma_0 \equiv \Sigma_\infty$. Before stating the result, we first fix some notation: let $\overline{\lambda} = \max\{\lambda_{1,m_0}, \lambda_{1,m_\infty}\}$ while we denote by $\underline{\lambda} = \min\{\lambda_{1,m_0}, \lambda_{1,m_\infty}\}$.

Corollary 4.1 *Assume that* $f(\lambda, x, s)$ *satisfies the hypotheses of Theorems 3.1 and 3.2. Suppose moreover that there exists* $\sigma > 0$ *such that*

$$f(\lambda, x, s) > \sigma\lambda s, \qquad \forall(\lambda, x, s) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}\,.$$

Then the sets Σ_0 *and* Σ_∞ *given respectively by Theorems 3.1 and 3.2 are equal, i.e.* $\Sigma_0 \equiv \Sigma_\infty$*. In particular, there exists at least one solution of* (P_λ) *for every* $\lambda \in (\underline{\lambda}, \overline{\lambda})$.

Moreover, assume that supercritical bifurcation at $\overline{\lambda}$ *and subcritical bifurcation at* $\underline{\lambda}$ *occur, then there exist* $\delta_1, \delta_2 > 0$ *such that if* $\lambda \in (\underline{\lambda} - \delta_1, \underline{\lambda})$ *or* $\lambda \in (\overline{\lambda}, \overline{\lambda} + \delta_2)$*, then* (P_λ) *possesses at least two solutions.*

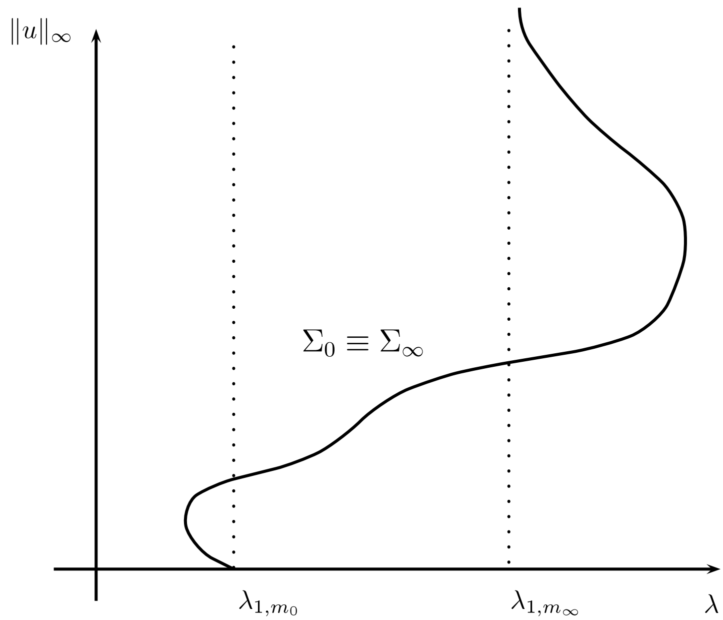

Figure 1: The case $\underline{\lambda} = \lambda_{1,m_0}$ and $\overline{\lambda} = \lambda_{1,m_\infty}$ in Corollary 4.1.

Remark 4.1 Notice that Theorems 4.1 and 4.2 provide sufficient conditions on the nonlinearity f for the subcriticality at $\underline{\lambda}$ and the supercriticality at $\overline{\lambda}$.

Proof. We recall that λ_{1,m_0} and λ_{1,m_∞} are respectively the only positive bifurcation points from 0 and ∞. Let also Σ_0 and Σ_∞ be given respectively by Theorems 3.1 and 3.2. If we prove that there exists $\lambda^* > 0$ such that problem (P_λ) does not posses any solution for $\lambda > \lambda^*$, then $\Sigma_0 \equiv \Sigma_\infty$. Indeed, in that case, since the continuum Σ_0 has to be unbounded, necessarily the projection of Σ_0 on the $\|u\|_\infty$–axis has to be unbounded and this means that $\Sigma_0 \equiv \Sigma_\infty$.

To show the existence of such a $\lambda^* > 0$, we assume that u is a nonzero solution of (P_λ). Taking ϕ_1 as a test function in (P_λ) (ϕ_1 is the function defined in (2.1)), we derive that

$$\lambda_1 \int_\Omega u\phi_1 = \int_\Omega f(\lambda, x, u)\phi_1 \geq \sigma\lambda \int_\Omega u\phi_1\,,$$

i.e. $\lambda \leq \frac{\lambda_1}{\sigma}$; we get a contradiction by choosing λ^* greater than $\frac{\lambda_1}{\sigma}$.

Hence $\Sigma_0 \equiv \Sigma_\infty$ is a continuum that connects λ_{1,m_0} with λ_{1,m_∞} and it follows that its projection contains the interval $(\underline{\lambda}, \overline{\lambda})$. Hence we deduce the existence of at least one solution if $\lambda \in (\underline{\lambda}, \overline{\lambda})$.

The multiplicity of solutions is a direct consequence of the subcriticality of the bifurcation from $\underline{\lambda}$ and the supercriticality of the bifurcation from $\overline{\lambda}$ (see Figure 1). □

Notice that the possibility that the two component are disjoint is also real. Indeed, we also give sufficient conditions in order to have $\Sigma_0 \cap \Sigma_\infty = \emptyset$.

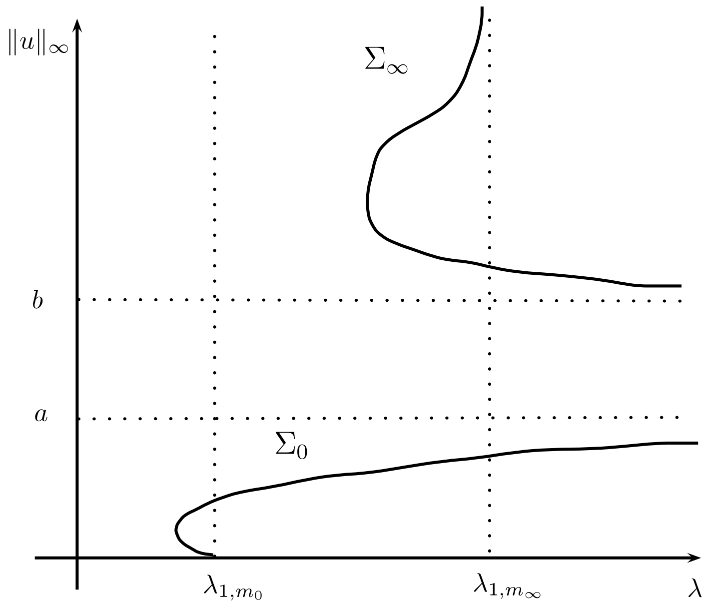

Figure 2: The case $\underline{\lambda} = \lambda_{1,m_0}$ and $\overline{\lambda} = \lambda_{1,m_\infty}$ in Corollary 4.2.

Corollary 4.2 *Suppose that the assumptions of Theorems 3.1 and 3.2 hold, and assume that there exists a bounded interval* $[a, b] \subset (0, \infty)$ *such that*

$$f(\lambda, x, s) < 0 \quad \forall (\lambda, x, s) \in \mathbb{R}^+ \times \Omega \times [a, b] .$$

Then there do not exist solutions u of (P_λ) *such that* $a < \|u\|_\infty < b$.

Moreover:

- *there exist at least one solution of* (P_λ) *if* $\underline{\lambda} < \lambda < \overline{\lambda}$ *and at least two solutions for any* $\lambda > \overline{\lambda}$*;*
- *if the bifurcation from* $\underline{\lambda}$ *is subcritical, there exist at least two solutions of* (P_λ) *if* $\underline{\lambda} - \delta_1 < \lambda < \underline{\lambda}$*, for a suitable* $\delta_1 > 0$*;*
- *if the bifurcation from* $\overline{\lambda}$ *is subcritical, there exist at least three solutions of* (P_λ) *if* $\overline{\lambda} - \delta_2 < \lambda < \overline{\lambda}$*, for a suitable* $\delta_2 > 0$.

Proof. The nonexistence of solutions u of (P_λ) such that $a < \|u\|_\infty < b$ is proved by contradiction. Indeed, if we assume that u is a solution of (P_λ) and $a < \|u\|_\infty < b$, we get a contradiction with the maximum principle (see Remark 2.2 for more details).

Hence the branch of solutions Σ_0 emanating from $(\lambda_{1,m_0}, 0)$ given by Theorem 3.1 is unbounded and cannot cross the value $\|u\|_\infty = a$. Hence $(\lambda_{1,m_0}, +\infty) \subseteq \mathrm{Proj}_\lambda(\Sigma_0)$ (the projection of Σ_0 on the λ–axis) and it follows that for any $\lambda > \lambda_{1,m_0}$ there exists a solution in Σ_0 of (P_λ) with norm less than a.

Analogously, the branch of solutions Σ_∞ given by Theorem 3.2 emanates from $(\lambda_{1,m_\infty}, \infty)$ and it cannot cross the value $\|u\|_\infty = b$. Hence, using the globality of Σ_∞ and since λ_{1,m_∞} is the unique bifurcation point from infinity, we deduce that $\mathrm{Proj}_\lambda(\Sigma_\infty) \supseteq (\lambda_{1,m_\infty}, +\infty)$. Hence, the existence of at least a solution in Σ_∞ of (P_λ) for any $\lambda > \lambda_{1,m_\infty}$ with norm bigger than b follows.

Finally, by the subcriticality and the supercriticality of the bifurcation points, we deduce the results concerning the multiplicity of solutions (see also Figure 2). □

Corollary 4.3 *Assume the hypotheses of Theorem 3.2 are satisfied. If* $f(\lambda, x, 0) \not\equiv 0$*, then problem* (P_λ) *has at least one solution if* $\lambda > \lambda_{1,m_\infty}$*.*

In addition, if (4.1) and (4.2) hold true, then there exists $\varepsilon > 0$ *such that the problem* (P_λ) *has at least two solutions for* $\lambda_{1,m_\infty} - \varepsilon < \lambda < \lambda_{1,m_\infty}$ *and, at least, one solution provided that* $\lambda \geq \lambda_{1,m_\infty}$*.*

Proof. By Theorem 3.2, we have a global bifurcation Σ_∞ from infinity at λ_{1,m_∞}. Moreover, since $f(\lambda, x, 0) \not\equiv 0$, we have no bifurcation from zero, which implies that the projection $\mathrm{Proj}_\lambda(\Sigma_\infty)$ has to be an unbounded interval in $(0, \infty)$. This show the existence of solution of (P_λ) for $\lambda > \lambda_{1,m_\infty}$.

Furthermore, if (4.1) and (4.2) hold true, then Theorem 4.1 applies and thus Σ_∞ is subcritical at λ_{1,m_∞}. This implies the existence of two solutions in Σ_∞ of (P_λ) for λ smaller than λ_{1,m_∞}, but sufficiently close to it and the existence of (at least) a solution of $(P_{\lambda_{1,m_\infty}})$. □

5 The antimaximum principle

In this final section we deal with the antimaximum principle. We prove for instance that, analogously to what happens in the case of the Laplacian operator, if h is a sufficiently smooth nonnegative function, then every solution u of the linear problem

$$(Q_\lambda) \qquad \begin{cases} (-\Delta)^{1/2}u = \lambda m(x)u + h(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

with m smooth and λ greater than $\lambda_{1,m}$ and sufficiently close to it, satisfies $u < 0$ in Ω.

We recall that $\lambda_{1,m}$, resp. $\varphi_{1,m} > 0$, is the first eigenvalue, resp. eigenfunction, to the associated eigenvalue problem (2.7). Precisely, we establish the following result.

Theorem 5.1 *Let* m, $h \in C^\gamma(\overline{\Omega})$ *for some* $\gamma \in (0,1)$*, with* $h = 0$ *on* $\partial\Omega$ *and* $m^+ \not\equiv 0$*. Then, there exists* $\varepsilon = \varepsilon(h) > 0$ *such that* (Q_λ) *has a unique solution* u_λ *for every* $\lambda \in (\lambda_{1,m} - \varepsilon, \lambda_{1,m} + \varepsilon)$*,* $\lambda \neq \lambda_{1,m}$*. Moreover*

1. If $\displaystyle\int_\Omega h\varphi_{1,m} < 0$*, then* *(a)* $u > 0$ *in* Ω *provided that* $\lambda_{1,m} < \lambda < \lambda_{1,m} + \varepsilon$.

(b) $u < 0$ *in* Ω *provided that* $\lambda_{1,m} - \varepsilon < \lambda < \lambda_{1,m}$.

2. *If* $\int_\Omega h\varphi_{1,m} > 0$, *then*

(a) $u < 0$ *in* Ω *provided that* $\lambda_{1,m} < \lambda < \lambda_{1,m} + \varepsilon$.

(b) $u > 0$ *in* Ω *provided that* $\lambda_{1,m} - \varepsilon < \lambda < \lambda_{1,m}$.

3. *If* $\int_\Omega h\varphi_{1,m} = 0$, *then any solution* (λ, u) *of* Q_λ *with* $\lambda \neq \lambda_{1,m}$ *changes sign in* Ω.

Proof. We prove the first statement. First of all, we notice that the problem is linear, then if $h \neq 0$, the Fredohlm Theorem states that for any $\lambda \neq \lambda_{k,m}$, $k \in \mathbb{N}$ there exists a solution u_λ of (Q_λ). Moreover, by a comparison principle, such a u_λ is the unique solution of (Q_λ).

Now we prove case a) of (1) (the proof of case b) follows similarly). Let us consider a sequence $\{\lambda_n\}$ strictly decreasing and converging to $\lambda_{1,m}$ and the sequence $\{u_n\}$ of solutions to (Q_{λ_n}). Clearly, $u_n \in L^\infty(\Omega)$, then we take $\varphi_{1,m}/\|u_n\|_\infty$ as a test function in (Q_{λ_n}) and we obtain

$$(\lambda_1 - \lambda_n)\int_\Omega \frac{u_n}{\|u_n\|_\infty} m(x)\varphi_{1,m} = \frac{1}{\|u_n\|_\infty}\int_\Omega h\varphi_{1,m} < 0. \tag{5.1}$$

We define $v_n = \frac{u_n}{\|u_n\|_\infty}$ which satisfy

$$\begin{cases} (-\Delta)^{1/2} v_n = \lambda_n m(x) v_n + \dfrac{h}{\|u_n\|_\infty} & \text{in } \Omega\,, \\ v_n = 0 & \text{on } \partial\Omega\,, \end{cases}$$

with $\|v_n\|_\infty = 1$. It is clear that, by Remark 3.1, there exists a subsequence (not relabeled) uniformly convergent to some v_0.Moreover, v_0 solves the problem

$$\begin{cases} (-\Delta)^{1/2} v_0 = \lambda_{1,m} m(x) v_0 & \text{in } \Omega\,, \\ v_0 = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Notice that it implies that either $v_0 = \varphi_{1,m}$ or $v_0 = -\varphi_{1,m}$. The information about the sign of v_0 is given by the sign of the right hand side in (5.1); precisely, since $\lambda_n > \lambda_{1,m}$ and $\int_\Omega h\varphi_{1,m} < 0$, we deduce that

$$\int_\Omega m(x) v_0 \varphi_{1,m} > 0\,,$$

and consequently $v_0 = \varphi_{1,m}$ necessarily.

By Proposition 2.1 we get that, for a subsequence (not relabeled), $v_n \to \varphi_{1,m}$ as $n \to \infty$ uniformly in $C^1(\overline{\Omega})$. Using the Hopf Lemma (see Lemma 4.3 in [10]), v_0 does not change sign and by the $C^1(\overline{\Omega})$ convergence we conclude that, for n sufficiently large, $u_n > 0$ in Ω.

Consequently, for any $\lambda_{1,m} < \lambda < \lambda_{1,m} + \varepsilon$ the solutions of (Q_λ) are strictly positive.

Finally, the proof of (2) follows in the same way while (3) is trivially deduced. □

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