COMPARISON RESULTS FOR UNBOUNDED SOLUTIONS FOR A PARABOLIC CAUCHY-DIRICHLET PROBLEM WITH SUPERLINEAR GRADIENT GROWTH

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ABSTRACT. In this paper we deal with uniqueness of solutions to the following problem

$$\begin{cases} u_t - \Delta_p u = H(t,x,\nabla u) & \text{in} \quad Q_T, \\ u(t,x) = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

where $Q_T = (0,T)\times\Omega$ is the parabolic cylinder, Ω is an open subset of $\mathbb{R}^N$, $N\geq 2$, $1<p<N$, and the right hand side $H(t,x,\xi):(0,T)\times\Omega\times\mathbb{R}^N\to\mathbb{R}$ exhibits a superlinear growth with respect to the gradient term.

1. Introduction

The present paper is devoted to the study of the uniqueness and, more in general, to the comparison principle between sub and supersolutions of nonlinear parabolic problems with lower order terms that have at most a power growth with respect to the gradient. More specifically, we set Ω a bounded open subset of $\mathbb{R}^N$, with $N\geq 3$, and $T>0$. We consider a Cauchy–Dirichlet problem of the type

$$\begin{cases} u_t - \Delta_p u = h(t,x,\nabla u) + f(t,x) & \text{in} \quad Q_T, \\ u(t,x) = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $Q_T = (0,T)\times\Omega$ denotes the parabolic cylinder, $-\Delta_p$ is the usual $p-$Laplacian with $p>1$, the functions u_0 and f belong to suitable Lebesgue spaces and $h(t,x,\xi)$ is a Cartheodory function that has (at most) $q-$growth with respect to the last variable, being q "superlinear" and smaller than p.

The model equation we have in mind is the following

$$\begin{cases} u_t - \Delta_p u = |\nabla u|^q + f(t,x) & \text{in} \quad Q_T, \\ u(t,x) = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

for $1<q<p$, $f\in L^r(0,T;L^m(\Omega))$, for some $r,m\geq 1$, and $u_0\in L^s(\Omega)$, for some $s\geq 1$.

The literature about comparison principles for weak sub/super solutions of (1.2) is mainly devoted to cases in which solutions are smooth (say for instance continuous), the equation is exactly the one in (1.2) or the growth of the nonlinear term is "sublinear".

Our aim is to generalize this kind of results to the case of *unbounded solutions* and *non regular data* (both the initial datum and the forcing term), dealing with sub/supersolutions in a suitable class.

Let us mention that in the elliptic framework such a kind of results have been studied in several papers using different techniques. Let us recall the papers [ABM], [BM] [BDMP], [BMMP], [BMMP2], [Me] (and references cited therein) where unbounded solutions for quasilinear equations have been treated. We want also to highlight the results of [Po2] (see also[BP], [LPR] and [LP]) that have inspired our work, where the comparison principle among unbounded sub/supersolutions has been proved, for sub/supersolutions that have a suitable power that belongs to the energy space.

Let us also mention that, as well explained in [ADP1] (see also [ADP2] for the parabolic counterpart) things change drastically when one deals with the so called *natural growth* (i.e. $q=p$ in (1.2)), since in this case the right class in which looking for uniqueness involves a suitable exponential of the solution (one can convince himself just by performing the Hopf-Cole transformation to the equation in (1.2)).

The literature is much poorer in the parabolic case, especially when unbounded solutions are considered. Let us mention the results in [F],[DNFG] where nonlinear problem of the type (1.1) are considered where $h(t,x,\xi)$ has a sublinear (in the sense of the p-Laplacian type operators, see [Ma] for more details about such a threshold) growth with respect to the last variable.

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In order to prove the comparison principle (that has the uniqueness has byproduct), several techniques have been developed. Let us mention, among the others, the results that have been proved by using the monotone rearrangement technique (see for instance [BDMP] and references cited therein) and by means of viscosity solutions (see for instance [CIL], [BDL] and references cited therein).

Our choice, that has been mainly inspired by [Po2], uses both an argument via linearization and a method that exploits a sort of convexity of the hamiltonian term with respect to the gradient. These two approaches are, in some sense, complementary since the first one (the linearization) works in the case $1 < p \leq 2$ while the second one (the "convex"one) deals with $p \geq 2$. Of course, the only case in which both of them are in force is when $p = 2$.

Since we want to deal with unbounded solutions and irregular data, the way of defining properly the sub/supersolutions is through the renormalized formulation (see [BM], [Pe], [BlP] and [PPP]).

The renormalized formulation, that is the most natural one in this framework, is helpful in order to face the first difficulty of our problem, that is the unboundedness of the sub/supersolutions. Indeed we can decompose the sub/supersolutions into their bounded part plus a reminder that can be estimated, using the uniqueness class we are working in.

According with the results in the stationary case, we prove that the uniqueness class (i.e. the class of functions for which we can prove the comparison principle, and uniqueness as a byproduct) is the set of functions whose a suitable power γ (that depends only on q, p and N) belongs to the energy space. Let us recall (see [Ma]) that such a class is also the right one in order to have existence of solutions.

Even more, we show, through a counterexample, that at least for $p = 2$, the class of uniqueness is the right one, adapting an argument of [BASW1]–[BASW2] to our case.

We first consider the case with $1 < p \leq 2$, and we look for an inequality solved by the difference between the bounded parts of the sub and supersolutions, using the linearization of the lower order term. Let us recall that this is the typical approach for singular (i.e. $p \leq 2$) operators, that has extensively used in several previous papers (see for instance [F] and [ABM] and references cited therein). In this case we are allowed to deal with general Leray-Lions operators, even if, due to a lack of regularity of the the sub/supersolutions, we cannot cover all the superlinear and subnatural growths.

The second part of the paper is devoted to the case $p \geq 2$ that is, in some way, more complicated, due to the degenerate nature of the operator. In fact, we need to straight the hypotheses on the differential operator considering a perturbation (through a matrix with bounded coefficients) of the standard p-Laplacian.

Here the idea is to perturb the difference between the bounded parts of the sub and supersolutions and to exploit the convexity of the lower order term with respect to the gradient (at least in the case $p = 2$, otherwise the general hypothesis is more involved).

The plan of the paper is the following: in Section 2 we collect all the statement of our results, while Section 3 is devoted to some technical results. The proofs of the main results are set in Section 4, if $1 < p \leq 2$ and in Section 5 if $p \geq 2$.

Finally in the Appendix there is an example that shows that the uniqueness class is the right one, at least for $p = 2$ and $1 < q < 2$.

2. Assumptions and statements of the results

As already explained in the Introduction, we deal with the following Cauchy-Dirichlet problem:

$$\begin{cases} u_t - \operatorname{div} a(t,x,\nabla u) = H(t,x,\nabla u) & \text{in} \quad Q_T, \\ u(t,x) = 0 & \text{on} \quad (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{2.1}$$

with $u_0 \in L^1(\Omega)$.

The main assumptions on the functions involved in (2.1) are the following: the vector valued function $a(t,x,\xi) : (0,T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that

$$\exists \alpha > 0 : \quad (a(t,x,\xi) - a(t,x,\eta)) \cdot (\xi - \eta) \geq \alpha(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2, \tag{2.2}$$

$$\exists \beta > 0 : \quad |a(t,x,\xi)| \leq \beta \left[\ell(t,x) + |\xi|^{p-1}\right], \quad \ell \in L^{p'}(Q_T), \tag{2.3}$$

$$a(t,x,0) = 0. \tag{2.4}$$

for a.e. $(t,x) \in Q_T$, $\forall \eta, \xi \in \mathbb{R}^N$, with $1 < p < N$.

As far as the lower order term is concerned, we suppose that $H(t,x,\xi) : (0,T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function that satisfies the following *growth condition*:

$$\exists c_1 > 0 : \quad |H(t,x,\xi)| \leq c_1 |\xi|^q + f \quad \text{with} \quad \max\left\{\frac{p}{2}, \frac{p(N+1) - N}{N+2}\right\} < q < p, \tag{2.5}$$

for a.e. $(t,x) \in Q_T$, $\forall \xi \in \mathbb{R}^N$ and with $f = f(t,x)$ belonging to some Lebesgue space.

First of all we need to determine the meaning of sub/supersolutions we want to deal with. Since we are interested in possibly irregular data and, in general, in unbounded solutions, the most natural way to mean sub/supersolutions is trough the renormalized formulation. In order to introduce such an issue, we first need to define a natural space where such sub/supersolutions are defined: taking inspiration from [BBGGPV], we set

$$\mathcal{T}_0^{1,p}(Q_T) = \left\{ u : Q_T \to \mathbb{R} \text{ a.e. finite : } T_k(u) \in L^p\big(0,T; W_0^{1,p}(\Omega)\big) \right\}, \quad \text{for } p \geq 1,$$

where $T_k(s) = \max\{-k, \min\{k,s\}\}$, for $k \geq 0$ and $s \in \mathbb{R}$.

Now we are ready to define the renormalized sub/super solutions to (2.1).

Definition 2.1. *We say that a function $u \in \mathcal{T}_0^{1,p}(Q_T)$ is a renormalized subsolution (respectively a supersolution) of (2.1) if*

$$H(t,x,\nabla u) \in L^1(Q_T), \quad u \in C([0,T]; L^1(\Omega))$$

and it satisfies:

$$\begin{aligned} -\int_\Omega S(u_0)\varphi(0)\,dx + \iint_{Q_T} [-S(u)\varphi_t + S'(u)a(t,x,\nabla u)\cdot\nabla\varphi + S''(u)a(t,x,\nabla u)\cdot\nabla u\,\varphi]\,dx\,dt \\ \leq (\geq) \iint_{Q_T} H(t,x,\nabla u)S'(u)\varphi\,dx\,dt, \end{aligned} \tag{2.6}$$

with

$$u(t,x)\big|_{t=0} \leq (\geq) u_0(x) \qquad \text{in } L^1(\Omega)\,,$$

for every $S \in W^{2,\infty}(\mathbb{R})$ such that $S'(\cdot)$ is nonnegative, compactly supported and for every

$$0 \leq \varphi \in L^\infty(Q_T)) \cap L^p(0,T; W_0^{1,p}(\Omega)),\ \varphi_t \in L^1(Q_T) + L^{p'}(0,T; W^{-1,p'}(\Omega)),\ \varphi(T,x) = 0.$$

Some remarks about the above definition are in order to be given.

Remark 2.2. *Let us observe that usually the renormalized formulation is equipped with an additional condition about the asymptotic behavior of the energy, i.e.*

$$\lim_{n\to\infty} \frac{1}{n} \iint_{\{n<|u|<2n\}} |\nabla u|^p\,dx\,dt = \lim_{n\to\infty} \frac{1}{n} \iint_{\{n<|v|<2n\}} |\nabla v|^p\,dx\,dt = 0\,. \tag{2.7}$$

Such a condition is required in order to guarantee that renormalized solution are, in fact, distributional ones. In our case we do not have to ask, in general, (2.7) to hold since it is a consequence of the class of uniqueness that we consider (see Lemma 3.4). More specifically, we have to impose such a condition only in the case in which we deal with L^1-data and with "low" values of q.

Remark 2.3. i) *Note that a subsolution (a supersolution) on Q_T turns out to be a subsolution (a supersolution) on Q_t for any $0 < t \leq T$. Thus, with an abuse of notation, we refer to Definition 2.1 even if we take into account (2.6) evaluated over Q_t, with $0 < t \leq T$.*

ii) *For renormalized solutions of an equation of the type (1.2), the regularity $u \in C([0,T]; L^1(\Omega))$ is deduced directly by the renormalized formulation, via a trace result (see [Po1]). However, since we are dealing with sub/supersolutions, we need to add it to the definition.*

2.1. **Assumptions for $p = 2$.** As already announced in the Introduction, for problem (2.1) with $p = 2$ we can use both the approach by linearization and the one by "convexity".

The first approach we want to deal with is the one by linearization. Hence we assume that $a(t,x,\xi) : (0,T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies (2.2)–(2.4), that in this particular case read as:

$$\exists \alpha > 0 : \quad (a(t,x,\xi) - a(t,x,\eta)) \cdot (\xi - \eta) \geq \alpha|\xi - \eta|^2 \tag{2.8}$$

$$\exists \beta > 0 : \quad |a(t,x,\xi)| \leq \beta\,[\ell(t,x) + |\xi|] \quad \text{for} \quad \ell \in L^2(Q_T), \tag{2.9}$$

$$a(t,x,0) = 0, \tag{2.10}$$

a.e. $(t,x) \in Q_T$, for all $\xi, \eta \in \mathbb{R}^N$.

Moreover we assume the growth assumption

$$\exists c_1 > 0 : \quad |H(t,x,\xi)| \leq c_1|\xi|^q + f(t,x) \quad \text{for} \quad 1 \leq q < 2, \tag{2.11}$$

a.e. in $(t,x) \in Q_T$, $\forall \xi \in \mathbb{R}^N$. In addition we suppose the following locally (weighted) Lipschitz condition

$$\exists c_2 > 0 : \quad |H(t,x,\xi) - H(t,x,\eta)| \leq c_2|\xi - \eta|\left[g(t,x) + |\xi|^{q-1} + |\eta|^{q-1}\right] \tag{2.12}$$

is in force, for some function $g(t,x)$ belonging to a suitable Lebesgue space we specify later.

Before stating our comparison results, we need to introduce the class of uniqueness. As for the elliptic case (see [Po2, BP] and also [GMP1]–[GMP2]), the right framework is the set of sub/supersolutions u, v whose power $\gamma = \gamma(q)$ belongs to the energy space for a suitable choice of γ. Moreover we consider the initial data u_0, v_0 belonging to $L^\sigma(\Omega)$ for some $\sigma \geq 1$. More precisely, we consider sub/supersolutions satisfying

$$u,\, v \in C([0,T]; L^\sigma(\Omega)) \qquad \text{with} \quad \sigma = \frac{N\big(q-1\big)}{2-q} \tag{2.13}$$

and

$$(1+|u|)^{\frac{\sigma}{2}-1}u,\, (1+|v|)^{\frac{\sigma}{2}-1}v \in L^2(0,T; H_0^1(\Omega)). \tag{2.14}$$

Such a class of uniqueness makes sense whenever $1 \leq \sigma$, i.e. if $q \geq 2 - \frac{N}{N+1}$.

Remark 2.4. *One can convince himself that the uniqueness class is the right one just by constructing a counterexample of a problem of the type (2.1) that admits (at least) two solutions, whose just one belongs to the right class. The construction of such a pair of solutions is a bit involved and we left it to the Appendix A.*

The assumptions about the data are strictly related to the value of the superlinearity q. For this reason, we split the superlinear growth of the gradient term into two subintervals for which, in turn, we require two different compatibility conditions on the data and two different class of uniqueness.

We first consider superlinear rates belonging to the range

$$2 - \frac{N}{N+1} < q \leq 2 - \frac{N}{N+2} \tag{2.15}$$

that correspond to the case $1 < \sigma \leq 2$, and that allows us to consider $f(t,x) \in L^r(0,T; L^m(\Omega))$ in (2.11) that verifies

$$m \neq \infty,\, r \neq \infty \quad \text{s.t.} \quad \frac{N\sigma}{m} + \frac{2\sigma}{r} \leq N + 2\sigma \tag{2.16}$$

while $g(t,x) \in L^d(Q_T)$ in (2.12) satisfies

$$g \in L^d(Q_T) \quad \text{with} \quad d = N+2. \tag{2.17}$$

Our first result is the following.

Theorem 2.5. *Assume that $a(t,x,\xi)$ satisfies (2.8)–(2.10), $H(t,x,\xi)$ satisfies (2.11) (2.12) and that (2.15)–(2.17) hold true. Let u and v be a renormalized subsolution and a supersolution of (2.1), respectively, satisfying (2.13), (2.14) and let u_0, $v_0 \in L^\sigma(\Omega)$ such that $u_0 \leq v_0$. Then $u \leq v$ in Q_T.*

Remark 2.6. *As far as the limit the case $q = 2 - \frac{N}{N-1}$ is concerned, we observe that the result of Theorem 2.5 still holds true assuming the data*

$$u_0 \in L^{1+\omega}(\Omega),\, f \in L^{1+\omega}(Q_T), \qquad \forall \omega > 0,$$

for sub/supersolutions u, v that belong to the class

$$(1+|u|)^{\frac{\omega-1}{2}}u,\, (1+|v|)^{\frac{\omega-1}{2}}v \in L^2(0,T; H_0^1(\Omega)).$$

The proof follows as the one of Theorem 2.5 with minor changes, so we omit it.

Secondly consider the range given by

$$1 \leq q < 2 - \frac{N}{N+1} \tag{2.18}$$

that correspond to $\sigma < 1$, and we require that the functions f and g satisfy

$$f \in L^1(Q_T) \tag{2.19}$$

and

$$g \in L^d(Q_T) \quad \text{with} \quad d > N+2. \tag{2.20}$$

Thus our result in this framework is the following.

Theorem 2.7. *Assume that $a(t,x,\xi)$ satisfies (2.8)–(2.10), $H(t,x,\xi)$ satisfies (2.11) (2.12) and that (2.18)–(2.20) hold true. Let u and v be a renormalized subsolution and a supersolution of (2.1), respectively, satisfying (2.7) and let u_0, $v_0 \in L^1(\Omega)$ such that $u_0 \leq v_0$. Then $u \leq v$ in Q_T.*

Let us observe that, in fact, our results do not cover all the interval $1 \le q < 2$. This is due to a lack of regularity of the sub/super solutions (see Remark 2.12 below).

The second approach to the comparison principle deals with a trick that uses the convexity of the lower order order term. Such a method is not as robust as the linearization one, so we need to strength the hypotheses on the differential operator.

We consider here the following problem

$$\begin{cases} u_t - \operatorname{div}\big(A(t,x)\nabla u\big) = H(t,x,\nabla u) & \text{in } Q_T, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega\,. \end{cases} \tag{2.21}$$

We assume that $A : (0,T)\times\Omega \to \mathbb{R}^{N\times N}$ is a bounded and uniformly elliptic matrix with measurable coefficients, i.e.

$$\begin{gathered} A(t,x) = \big\{a_{ij}(t,x)\big\}_{i,j=1}^{N} \quad \text{with} \quad a_{ij} \in L^\infty(Q_T) \quad \forall 1 \le i,j \le N, \\ \text{such that} \quad \exists \alpha,\,\beta: \quad 0 < \alpha \le \beta \quad \text{and} \quad \alpha|\xi|^2 \le A(t,x)\xi\cdot\xi \le \beta|\xi|^2\,, \end{gathered} \tag{2.22}$$

for almost every $(t,x) \in Q_T$ and for every $\xi \in \mathbb{R}^N$,

As far as the lower order term is concerned, we suppose that the nonlinear term $H(t,x,\xi)$ satisfies (2.11) with $1 < q < 2$ and it can be decomposed as

$$H(t,x,\xi) = H_1(t,x,\xi) + H_2(t,x,\xi) \tag{2.23}$$

where, for a.e. $(t,x) \in Q_T$ and for every ξ, η in $\mathbb{R}^N$, the functions $H_1(t,x,\xi)$ and $H_2(t,x,\xi)$ verify:

- $H_1(t,x,\xi) : (0,T)\times\Omega\times\mathbb{R}^N \to \mathbb{R}$ is a convex function with respect to the ξ variable, i.e.

$$\forall \varepsilon \in (0,1) \quad H_1\left(t,x,\varepsilon\xi + (1-\varepsilon)\eta\right) \le \varepsilon H_1(t,x,\xi) + (1-\varepsilon)H_1(t,x,\eta) \quad \forall \xi,\eta \in \mathbb{R}^N\,; \tag{2.24}$$

- $H_2(t,x,\xi) : (0,T)\times\Omega\times\mathbb{R}^N \to \mathbb{R}$ is a Lipschitz function with respect to ξ, namely

$$\exists c_2 > 0: \quad |H_2(t,x,\xi) - H_2(t,x,\eta)| \le c_2|\xi-\eta| \tag{2.25}$$

  that satisfies the following inequality for sufficiently small $\varepsilon > 0$

$$H_2(t,x,(1-\varepsilon)\xi) - (1-\varepsilon)H_2(t,x,\xi) \le 0 \tag{2.26}$$

  for almost every $(t,x) \in Q_T$ and for all ξ, $\eta \in \mathbb{R}^N$.

As for the approach by linearization, we have two types of results, depending on the regularity of the sub/supersolutions under consideration.

First we deal with solutions in the class (2.13)–(2.14): in this case we consider lower order terms whose growth with respect to the gradient is at most a power q in the range

$$2 - \frac{N}{N+1} < q < 2\,, \tag{2.27}$$

and we assume that f in (2.11) belongs to $L^r(0,T;L^m(\Omega))$ with (m,r) such that (2.16) holds true.

Hence we have the following result.

Theorem 2.8. *Assume that $A(t,x)$ satisfies (2.22) and $H(t,x,\xi)$ (2.11) together with (2.23)–(2.26), (2.27) and (2.16), and let u and v be, respectively, a renormalized subsolution and a supersolution of (2.21) satisfying (2.13)–(2.14). and let u_0, $v_0 \in L^\sigma(\Omega)$ be such that $u_0 \le v_0$. Then we have that $u \le v$ in Q_T.*

As far as the low values of q are considered, we deal with the same range considered in (2.18) and L^1 data.

Theorem 2.9. *Assume that $A(t,x)$ satisfies (2.22) and $H(t,x,\xi)$ (2.11) together with (2.23)–(2.26), (2.18) and (2.19). Let u and v be a renormalized subsolution and a supersolution of (2.21), respectively, satisfying (2.7) and let u_0, $v_0 \in L^1(\Omega)$ with $u_0 \le v_0$. Then we have that $u \le v$ in Q_T.*

2.2. **Assumptions for $1<p<2$.** Le us now go back to our original problem (2.1) and let us assume that the vector valued function $a(t,x,\xi)$ satisfies (2.2)–(2.4), with $1<p<2$.

As far as the lower order term is concerned, we suppose that it satisfies the *growth condition* (2.5) and we assume that a suitable weighted Lipschitz assumption with respect to the last variable is in force, i.e.

$$\exists c_2>0: \quad |H(t,x,\xi)-H(t,x,\eta)|\le c_2|\xi-\eta|\left[g+|\xi|^{q-1}+|\eta|^{q-1}\right] \tag{2.28}$$

for a.e. $(t,x)\in Q_T$, for all $\xi,\eta\in\mathbb{R}^N$ and for some measurable function $g=g(t,x)$ belonging to $L^d(Q_T)$, for a suitable choice of $d\ge 1$.

In this setting, we determine two ranges of q each of them giving a different type of result in function of the required class of uniqueness (and the regularity) of the solutions.

We start by considering

$$\begin{gathered} p-\frac{N}{N+1}<q\le p-\frac{N}{N+2} \quad \text{for} \quad 1+\frac{N}{N+1}<p<2 \\ \text{and} \\ 1\le q<p-\frac{N}{N+2} \quad \text{for} \quad 1+\frac{N}{N+2}\le p<1+\frac{N}{N+1} \end{gathered} \tag{2.29}$$

together with

$$g\in L^d(Q_T) \quad \text{with} \quad d=\frac{N(q-(p-1))-p+2q}{q-1}. \tag{2.30}$$

Moreover we assume that

$$\begin{gathered} f\in L^r(0,T;L^m(\Omega)) \quad \text{with} \quad (m,r) \quad \text{such that} \\ m\neq\infty,\ r\neq\infty, \quad \frac{N\sigma}{m}+\frac{N(p-2)+p\sigma}{r}\le N(p-1)+p\sigma. \end{gathered} \tag{2.31}$$

When we deal with this range, we assume the continuity regularity

$$u,\, v\in C([0,T];L^\sigma(\Omega)) \quad \text{with} \quad \sigma=\frac{N(q-(p-1))}{p-q} \tag{2.32}$$

and we deal with the uniqueness class

$$(1+|u|)^{\gamma-1}u,\,(1+|v|)^{\gamma-1}v\in L^p(0,T;W_0^{1,p}(\Omega)) \quad \text{with} \quad \gamma=\frac{\sigma+p-2}{p}. \tag{2.33}$$

In this ranges of values of q we have that $\sigma\in(1,2)$.

Theorem 2.10. *Assume (2.2)–(2.4), and that $H(t,x,\xi)$ satisfies (2.5), (2.28) with (2.29)–(2.31). Let u and v be a subsolution and a supersolution of (2.1), respectively, satisfying (2.32)–(2.33). Then $u\le v$ in Q_T.*

The next range corresponds to the case of lower values of q. Namely, we consider

$$1\le q<p-\frac{N}{N+1} \quad \text{and} \quad 1+\frac{N}{N+1}<p<2 \tag{2.34}$$

and that g fulfils

$$g\in L^d(Q_T) \quad \text{with} \quad d>\frac{p(N+1)-N}{p(N+1)-(2N+1)} \quad \left(d=\infty \text{ if } p=1+\frac{N}{N+1}\right). \tag{2.35}$$

As far as the source term f is concerned, we suppose

$$f\in L^1(Q_T). \tag{2.36}$$

Thus the result is the following.

Theorem 2.11. *Assume (2.2)–(2.3) and that $H(t,x,\xi)$ satisfies (2.5) and (2.28) with (2.34)–(2.36). Let u and v be a renormalized subsolution and a renormalized supersolution of (2.1), respectively, such that (2.7) holds true. Then, we have that $u\le v$ in Q_T.*

Remark 2.12. *Let us mention some peculiarity of our results, for $1<p\le 2$.*

i) *It is worth pointing out that the case $p-\frac{N}{N+2}<q<p$ is not considered here. Indeed, as already observed in the elliptic case (see [Po2, Remark 3.5]), we would need to require more regularity on the gradient of the sub/supersolutions, in order to apply the linearization technique, which turns out to be unnatural in our framework. Indeed we should require the sub/supersolution to belong to the space*

$$L^\mu(0,T;W_0^{1,\mu}(\Omega)) \qquad \textit{with} \qquad \mu=(\sigma-1)(p-q)+q=N(q-(p-1))+2q-p$$

(see Lemma 3.4) and $\mu > p$ if $p - \frac{N}{N+2} < q$. Consequently we would have such a result under the additional assumption that $u \in L^\mu(0,T;W_0^{1,\mu}(\Omega))$.

ii) *Note that the critical growth $q = p - \frac{N}{N+1}$, that corresponds to the case $\sigma = 1$ and $m = r = 1$, has been excluded in (2.29) ((2.15) if $p = 2$). For such a value we have a slightly different result whose proof follows from the one of Theorem 2.11, with the following hypotheses on the data: $u_0 \in L^{1+\omega}(\Omega)$ and $f \in L^{1+\omega}(Q_T)$ with $\omega > 0$.*

2.3. **Assumptions for $2 < p < N$.** In this case we change (2.21) into the following problem:

$$\begin{cases} u_t - \operatorname{div}\big(A(x)|\nabla u|^{p-2}\nabla u\big) = H(t,x,\nabla u) & \text{in } \ Q_T, \\ u(t,x) = 0 & \text{on } \ (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \ \Omega, \end{cases} \tag{2.37}$$

where the matrix $A(x)$ is bounded, coercive and with measurable coefficients, while the right hand side satisfies a superlinear growth condition with respect to the gradient.

More precisely, we assume that $A : \Omega \to \mathbb{R}^{N\times N}$ is a bounded and uniformly elliptic matrix with measurable coefficients, i.e.

$$\begin{gathered} A(x) = \{a_{ij}(x)\}_{i,j=1}^N \quad \text{with} \quad a_{ij} \in L^\infty(\Omega) \quad \forall 1 \le i,j \le N\,, \\ \text{such that} \quad \exists \alpha, \beta:\ 0 < \alpha \le \beta \quad \text{and} \quad \alpha|\xi|^2 \le A(x)\xi\cdot\xi \le \beta|\xi|^2 \quad \text{a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^N. \end{gathered} \tag{2.38}$$

As far as the Hamiltonian term is concerned, we suppose, in addition to (2.5), that $\exists M > 0$, such that for any $\varepsilon \in (0,1)$

$$\begin{gathered} H(t,x,\xi) - (1-\varepsilon)^{p-1} H\bigg((1-\varepsilon)^{p-2}\ t, x, \frac{\eta}{1-\varepsilon}\bigg) \le c_2 \varepsilon^{1-q}|\xi-\eta|^q + L|\xi-\eta|\left(|\xi|^{\frac{p-2}{2}} + |\eta|^{\frac{p-2}{2}}\right) + \varepsilon M \\ \text{with} \quad p-1 < q < p, \end{gathered} \tag{2.39}$$

a.e. $(t,x) \in Q_T$, for all ξ, $\eta \in \mathbb{R}^N$ with $\xi \ne \eta$ and $\varepsilon \in (0,1)$.

The above hypothesis seems to be quite technical, since it combines several properties of the nonlinear lower order term. It is not so hard to see that, for example, the model Hamiltonian

$$H(t,x,\xi) = |\xi|^q + f_0(t,x)$$

satisfies (2.39) for $p-1 < q < p$ and for a function f_0 bounded above and not increasing with respect to the t variable. Let us underline that also some perturbations, through locally Lipschitz function, weighted with the $(p-2)/2$ power of the gradient of such a model Hamiltonian still fulfill hypothesis (2.39).

As for the previous results, we have two regimes depending on the values of q.

We first deal with the range

$$p - \frac{N}{N+1} < q < p \tag{2.40}$$

that let us considering solutions in the class (2.32)–(2.33).

Theorem 2.13. *Assume (2.38) and that $H(t,x,\xi)$ satisfies (2.5) together with (2.31), (2.39) and (2.40). Let u and v be a subsolution and a supersolution of (2.37), respectively, satisfying (2.32) and (2.33). Assume that $u_0 \le v_0 \le \bar{v}_0 < \infty$ with u_0, $v_0 \in L^\sigma(\Omega)$. Then we have that $u \le v$ in Q_T.*

Next, we consider the last case, that is the range

$$p - 1 < q < p - \frac{N}{N+1} \tag{2.41}$$

and we have the following result.

Theorem 2.14. *Assume (2.38) and that $H(t,x,\xi)$ satisfies (2.5) together with (2.36), (2.39) and (2.41). Let u and v be a subsolution and a supersolution of (2.37), respectively, such that (2.7) holds. Assume that $u_0 \le v_0 \le \bar{v}_0 < \infty$ with u_0, $v_0 \in L^\sigma(\Omega)$. Then we have that $u \le v$ in Q_T.*

3. Notation and basic tools

With the purpose of dealing with the bounded part of the sub/supersolutions considered during the paper, we here introduce a smooth approximation of the classical truncation function $T_n(s) = \max\{-n, \min\{n, s\}\}$.

We define the smoothed truncation function $S_n(\cdot)$ and $\theta_n(\cdot)$ as follows:

$$S_n(z)=\int_0^z \theta_n(\tau)\,d\tau, \qquad \text{while} \qquad \theta_n(z)=\begin{cases} 1 & |z|\le n,\\ \dfrac{2n-|z|}{n} & n<\ |z|\le 2n,\\ 0 & |z|>2n. \end{cases} \tag{3.1}$$

Moreover, here and in all the paper, we denote by $G_k(z)$ the function

$$G_k(z)=(z-k)_+$$

for every $z\in\mathbb{R}$ and for any $k\ge 0$.

Here we recall a classical parabolic regularity result that we use systematically in the following.

Theorem 3.1 (Gagliardo-Nirenberg). *Let $\Omega\subset\mathbb{R}^N$ be a bounded and open subset and $T>0$; if*

$$w\in L^\infty(0,T;L^h(\Omega))\cap L^\eta(0,T;W_0^{1,\eta}(\Omega))$$

with

$$h,\,\eta\ge 1,\ \eta<N \quad \text{and} \quad h\le \eta^*\,,$$

then we have that

$$w\in L^y(0,T;L^j(\Omega))$$

where the pair (j,y) fulfils

$$h\le j\le \eta^*,\ \eta\le y\le\infty \qquad \text{and} \qquad \frac{Nh}{j}+\frac{N(\eta-h)+h\eta}{y}=N.$$

Moreover there exists a positive constant $c(N,\eta,h)$ such that the following inequality holds true:

$$\int_0^T \|w(t)\|_{L^j(\Omega)}^y\,dt\le c(N,\eta,h)\|w\|_{L^\infty(0,T;L^h(\Omega))}^{y-\eta}\int_0^T\|\nabla w(t)\|_{L^\eta(\Omega)}^\eta\,dt. \tag{3.2}$$

Next we state two useful Lemmata that we will use in the sequel in order to conclude the proofs of our results.

Lemma 3.2. *Let $w\in L^\rho(0,T;W_0^{1,\rho}(\Omega))\cap L^\infty(0,T;L^\nu(\Omega))$, with $\rho,\,\nu\ge 1$ satisfy*

$$\exists c_0>0,\ \eta\ge 1 \ \text{ and } \ m>0:\quad \|G_k(w)\|_X\le c_0\|G_k(w)\|_X^\eta\left(\iint_{E_k}B\,dx\,dt\right)^m \quad \text{for some} \quad B\in L^1(Q_T),$$

for any $k\ge k_0$, where X is a Banach space, and $E_k=\{(t,x)\in Q_T:\ w>k\,,\ |\nabla w|>0\}$. Then $w\le 0$ in Q_T.

Proof. The proof follows direclty from the one of Lemma 2.1 in [Po2]. □

The next Lemma is a sort of parabolic version of the above one.

Lemma 3.3. *Let $w\in L^p(0,T;W_0^{1,p}(\Omega))\cap C([0,T];L^\sigma(\Omega))$ be a function satisfying the following inequality:*

$$\sup_{s\in(0,t)}\|w(s)\|_{L^\sigma(\Omega)}^\sigma+\|w\|_{L^p(0,t;W_0^{1,p}(\Omega))}^p\le c_0\sup_{s\in(0,t)}\|w(s)\|_{L^\sigma(\Omega)}^\eta\|w\|_{L^p(0,t;W_0^{1,p}(\Omega))}^m \tag{3.3}$$

with $0\le t\le T$ and for some $c_0>0$, $\eta>0$, $m\ge p$ and with $\|w(0)\|_{L^\sigma(\Omega)}=0$. Then $w\equiv 0$ in Q_T.

Proof. Without loss of generality we take into account the case $m=p$. Then, since $\|w(0)\|_{L^\sigma(\Omega)}=0$ and by the continuity assumption we define

$$T^*=\sup\left\{\tau>0:\quad \|w(s)\|_{L^\sigma(\Omega)}^\sigma\le\frac{1}{(2c_0)^{\frac{\sigma}{\eta}}}\quad \forall s\le\tau\right\}>0$$

which implies that, at least for $s\le T^*$, we get

$$\sup_{s\in(0,t)}\|w(s)\|_{L^\sigma(\Omega)}^\sigma+\frac{1}{2c_0}\|w\|_{L^p(0,t;W_0^{1,p}(\Omega))}^p\le 0 \tag{3.4}$$

from (3.3). Now, let us suppose by contradiction that $T^*<T$. Then, if $t=T^*$ and by definition of T^* we would find $\|w(T^*)\|_{L^\sigma(\Omega)}^\sigma=\frac{1}{(2c_0)^{\frac{\sigma}{\eta}}}\le 0$ which is in contrast with the assumption $c_0>0$. We thus deduce that (3.4) holds for all $t\le T$ and, in particular, we conclude that $\|w(t)\|_{L^\gamma(\Omega)}\equiv 0$ for every $t\in[0,T]$. □

During the proof of our main results, we need some regularity results, as the next two Lemmata.

Lemma 3.4. *Let $u \in C([0,T];L^\sigma(\Omega))$ be such that (2.33) holds true. Then*

$$|\nabla u| \in L^{N(q-(p-1))+2q-p}(Q_T) \quad \text{with} \quad 1 \le \sigma \le 2 \quad \left(i.e.\ \frac{N}{N+2} \le p-q \le \frac{N}{N+1}\right) \tag{3.5}$$

and

$$\frac{1}{n}\iint_{\{n<|u|<2n\}} |\nabla u|^p \, dx\, dt = o(n^{-(\sigma-1)}) \qquad \text{as } n \to \infty. \tag{3.6}$$

Proof. Let us start with the proof of the regularity in (3.5); using (2.33) and Theorem 3.1 with $\eta = p$ and $h = \frac{\sigma}{\gamma}$ we deduce that $u \in L^{p\frac{N\gamma+\sigma}{N}}(Q_T)$. Then, by Young's inequality, we get

$$|\nabla u|^{p\frac{N\gamma+\sigma}{N+\sigma}} \le c\frac{|\nabla u|^p}{(1+u)^{p(\gamma-1)}} + c(1+u)^{p\frac{N\gamma+\sigma}{N}}$$

and (3.5) follows by definitions of σ and γ.

As far as (3.6) is concerned, we have the inequality

$$\frac{n^{\sigma-1}}{n}\iint_{\{n<|u|<2n\}} |\nabla u|^p \, dx\, dt \le c_\gamma \iint_{\{n<|u|<2n\}} |\nabla |u|^\gamma|^p \, dx\, dt\,.$$

Then, since $n \ge 1$ and consequently dealing with $|u| > 1$, it implies that we can employ (2.33) and thus $\text{meas}\{(t,x) \in (0,T)\times\Omega : n < |u| \le 2n\} \to 0$ as $n \to \infty$ and thus (3.6) follows. □

Lemma 3.5. *Let u, v be respectively a renormalized subsolution and a renormalized supersolution of (2.1) such that (2.7), (2.5) with (2.34) hold. Then*

$$|\nabla u_+|,\, |\nabla v_-| \in L^r(Q_T) \quad \text{for} \quad 1 \le r < p - \frac{N}{N+1}. \tag{3.7}$$

Proof. We only deal with the case of subsolution u, since having v be a supersolution implies that $-v$ is a subsolution.

In fact, (3.7) is a consequence of Corollary 4.7 (see also Remark 4.2) applied to the positive part of u, that yields to the following inequality

$$\int_\Omega \Theta(u_+(T))\, dx + \alpha \iint_{Q_T} |\nabla T_k(u_+)|^p \, dx\, dt \le k\left[\|H(t,x,\nabla u)\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)}\right].$$

By the standard results on the regularity, we deduce that $u_+ \in L^r(0,T;W^{1,r}(\Omega))$ for every $r < p - \frac{N}{N+1}$ (see [ST] and [BDGO]). □

We conclude this Section with another useful result.

Lemma 3.6. *Let ρ, $m \ge 1$ and $w \in \mathcal{T}_0^{1,\rho}(Q_T) \cap L^\infty(0,T;L^m(\Omega))$ satisfy*

$$\|w\|^m_{L^\infty(0,T;L^m(\Omega))} \le L \qquad \text{and} \qquad \iint_{Q_T} \frac{|\nabla w|^\rho}{(|w|+\mu)^\gamma}\, dx\, dt \le M\mu^{-\nu} \tag{3.8}$$

where $L,\ M,\ \mu,\ \nu > 0$ and $0 < \gamma < \rho$. Then

$$\exists c = c(\rho,\gamma,N,\mu,m) : \qquad \||\nabla w|^b\|_{L^1(Q_T)} \le c\left(L^{\frac{\gamma-\nu}{N+m}}M\right)^{\frac{b(N+m)}{N(\rho-\gamma+\nu)+m\rho}} \quad \text{with} \quad b = \frac{N(\rho-\gamma)+m\rho}{N+m}. \tag{3.9}$$

Proof. The above assumptions on w imply that we can apply Theorem 3.1 to the function

$$\left((|T_k(w)|+\mu)^{\frac{\rho-\gamma}{\rho}} - \mu^{\frac{\rho-\gamma}{\rho}}\right) \in L^\infty(0,T;L^{\frac{m\rho}{\rho-\gamma}}(\Omega)) \cap L^\rho(0,T;W_0^{1,\rho}(\Omega))$$

and we get the regularity estimate

$$\begin{aligned} &\iint_{Q_T} \left((|T_k(w)|+\mu)^{\frac{\rho-\gamma}{\rho}} - \mu^{\frac{\rho-\gamma}{\rho}}\right)^{\rho\frac{N+\frac{m\rho}{\rho-\gamma}}{N}} dx\, dt \\ &\le \left\|(|T_k(w)|+\mu)^{\frac{\rho-\gamma}{\rho}} - \mu^{\frac{\rho-\gamma}{\rho}}\right\|^{\frac{m\rho^2}{N(\rho-\gamma)}}_{L^\infty(0,T;L^{\frac{m\rho}{\rho-\gamma}}(\Omega))} \iint_{Q_T} \frac{|\nabla T_k(w)|^\rho}{(|T_k(w)|+\mu)^\gamma}\, dx\, dt. \end{aligned} \tag{3.10}$$

Then, since the inequality $(a+b)^\alpha \le a^\alpha + b^\alpha$ holds for a, $b > 0$, $0 < \alpha < 1$ and $\frac{\rho-\gamma}{\rho} < 1$, we have by (3.10), combined with (3.8), and Fatou's Lemma

$$\iint_{Q_T} \left((|w|+\mu)^{\frac{\rho-\gamma}{\rho}}\right)^{\rho\frac{N+\frac{m\rho}{\rho-\gamma}}{N}} dx\, dt \le c\left(\mu^{\frac{N(\rho-\gamma)+m\rho}{N}} + L^{\frac{\rho}{N}}M\mu^{-\nu}\right).$$

Minimizing the right hand side above with respect to μ, we get that the minimum is achieved at $\mu = c\left(L^{\frac{\rho}{N}}M\right)^{\frac{N}{N(\rho-\gamma+\nu)+m\rho}}$ for a constant c that depends only on ρ, γ, N, ν, m.
We are now ready to prove (3.9): we use the Hölder's inequality with $(\frac{\rho}{b}, \frac{\rho}{\rho-b})$ in order to get

$$\iint_{Q_T} |\nabla w|^b \,dx\,dt \le \left(\iint_{Q_T} \frac{|\nabla w|^\rho}{(|w|+\mu)^\gamma}\,dx\,dt\right)^{\frac{b}{\rho}} \left(\iint_{Q_T} (|w|+\mu)^{\frac{\gamma b}{\rho-b}}\,dx\,dt\right)^{\frac{\rho-b}{\rho}}$$

and we use that b satisfies $\frac{\gamma b}{\rho-b} = \frac{N(\rho-\gamma)+m\rho}{N}$, so that (3.9) follows from the choice of μ. □

4. Proofs in the case $1 < p \le 2$

We start by proving Theorem 2.10 and Theorem 2.5.

Proof of Theorem 2.10 and of Theorem 2.5. As anticipated, we need to rewrite the inequalities satisfied by sub/supersolutions in terms of their bounded parts plus some (quantified) reminder. To this aim, we set

$$u_n = S_n(u) \quad \text{and} \quad v_n = S_n(v) \tag{4.1}$$

where $S_n(\cdot)$ has been defined in (3.1). We consider the renormalized formulation in (2.6) with $S(u) = \theta_n(u)$ so we obtain

$$\begin{aligned}\int_\Omega u_n(t)\varphi(t)\,dx &+ \iint_{Q_t} a(s,x,\nabla u)\theta_n(u)\cdot\nabla\varphi + a(s,x,\nabla u)\cdot\nabla u\,\theta_n'(u)\varphi\,dx\,ds \\ &\le \iint_{Q_t} H(s,x,\nabla u)\theta_n(u)\varphi\,dx\,ds + \int_\Omega u_n(0)\varphi(0)\,dx.\end{aligned}$$

Reasoning in the same way on the supersolution v - of course, with $S(v) = \theta_n(v)$ - and considering the difference between the above inequalities, we get $\forall 0 < t \le T$,

$$\begin{aligned}&\int_\Omega (u_n(t)-v_n(t))\varphi(t)\,dx \\ &+ \iint_{Q_t} [a(s,x,\nabla u)\theta_n(u) - a(s,x,\nabla v)\theta_n(v)]\cdot\nabla\varphi + [a(s,x,\nabla u)\cdot\nabla u\,\theta_n'(u) - a(s,x,\nabla v)\cdot\nabla v\,\theta_n'(v)]\,\varphi\,dx\,ds \\ &\le \iint_{Q_t} [H(s,x,\nabla u)\theta_n(u) - H(s,x,\nabla v)\theta_n(v)]\,\varphi\,dx\,ds + \int_\Omega [u_n(0)-v_n(0)]\varphi(0)\,dx.\end{aligned}$$

We now define

$$z_n = u_n - v_n \tag{4.2}$$

and rewrite the above inequality as

$$\begin{aligned}\int_\Omega z_n(t)\varphi(t)\,dx &+ \iint_{Q_t} [a(s,x,\nabla u_n) - a(s,x,\nabla v_n)]\cdot\nabla\varphi\,dx\,ds \\ &\le \iint_{Q_t} [H(s,x,\nabla u_n) - H(s,x,\nabla v_n)]\,\varphi\,dx\,ds + R_n\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}R_n = &\iint_{Q_t} [a(s,x,\nabla u_n) - a(s,x,\nabla v_n) - a(s,x,\nabla u)\theta_n(u) + a(s,x,\nabla v)\theta_n(v)]\cdot\nabla\varphi\,dx\,ds \\ &- \iint_{Q_t} [a(s,x,\nabla u)\cdot\nabla u\,\theta_n'(u) - a(s,x,\nabla v)\cdot\nabla v\,\theta_n'(v)]\,\varphi\,dx\,ds \\ &+ \iint_{Q_t} [H(s,x,\nabla u)\theta_n(u) - H(s,x,\nabla v)\theta_n(v) - H(s,x,\nabla u_n) + H(s,x,\nabla v_n)]\,\varphi\,dx\,ds \\ &+ \int_\Omega z_n(0)\varphi(0)\,dx.\end{aligned}$$

In virtue of the density result [PPP, Proposition 4.2], we are allowed to take

$$\varphi(z_n) = \left[(G_k(z_n)+\mu)^{\sigma-1}\right] - \mu^{\sigma-1}, \quad \mu > 0,$$

in the inequality in (4.3) and then, thanks also to (2.2) and (2.28), we obtain

$$\begin{aligned}&\int_\Omega \Phi_k(z_n(t))\,dx + \alpha(\sigma-1)\iint_{Q_t} \left||\nabla u_n|^2 + |\nabla v_n|^2\right|^{\frac{p-2}{2}} \frac{|\nabla G_k(z_n)|^2}{(G_k(z_n)+\mu)^{2-\sigma}}\,dx\,ds \\ &\le c_1 \iint_{Q_t} |\nabla G_k(z_n)|\left[g + |\nabla u_n|^{q-1} + |\nabla v_n|^{q-1}\right]\left(\left[(G_k(z_n)+\mu)^{\sigma-1}\right] - \mu^{\sigma-1}\right)\,dx\,ds + R_n\end{aligned} \tag{4.4}$$

where $\Phi_k(z)\,dx = \int_0^{G_k(z)} \varphi(\tau)\,d\tau$.

We want to prove now that $\lim_{n\to\infty} R_n = 0$. From now on, we denote by ω_n any quantity that vanishes as n diverges, and we set

$$R_n = I_1 + I_2 + I_3 + I_4 \tag{4.5}$$

where

$$\begin{aligned}
I_1 &= \iint_{Q_t} \big[[a(s,x,\nabla u_n) - a(s,x,\nabla v_n)] - [\theta_n(u)a(s,x,\nabla u) - \theta_n(v)a(s,x,\nabla v)] \big] \cdot \nabla z_n \varphi'(z_n)\, dx\, ds,\\
I_2 &= -\iint_{Q_t} [\theta_n'(u)a(s,x,\nabla u)\cdot\nabla u - \theta_n'(v)a(s,x,\nabla v)\cdot\nabla v]\,\varphi(z_n)\, dx\, ds,\\
I_3 &= \iint_{Q_t} \big[[\theta_n(u)H(s,x,\nabla u) - \theta_n(v)H(s,x,\nabla v)] - [H(s,x,\nabla u_n) - H(s,x,\nabla v_n)] \big]\varphi(z_n)\, dx\, ds\\
I_4 &= \int_\Omega z_n(0)\varphi(0)\, dx.
\end{aligned} \tag{4.6}$$

Let us start by studying I_1. The definition of $\theta_n(\cdot)$ and (2.4) imply that

$$\begin{aligned}
I_1 = &\iint_{\{n<|u|<2n\}} \big[a(s,x,\nabla u_n) - \theta_n(u)a(s,x,\nabla u)\big]\cdot\nabla(u_n - v_n)\varphi'(u_n - v_n)\, dx\, ds\\
&- \iint_{\{n<|v|<2n\}} \big[a(s,x,\nabla v_n)) - \theta_n(v)a(s,x,\nabla v)\big]\cdot\nabla(u_n - v_n)\varphi'(u_n - v_n)\, dx\, ds
\end{aligned} \tag{4.7}$$

being $a(s,x,\nabla u)\theta_n(u) - a(s,x,\nabla u_n) \equiv 0$ when $|u| \le n$ and since $\nabla u_n \equiv 0$ when $|u| \ge 2n$ (and the same hols for v_n). We just prove that the first integral in (4.7) behaves as ω_n since the second one can be dealt in the same way. The definition of $\varphi(\cdot)$ and (2.3) allow us to deduce that

$$\begin{aligned}
&\iint_{\{n<|u|<2n\}} \big[a(s,x,\nabla u_n) - \theta_n(u)a(s,x,\nabla u)\big]\cdot\nabla(u_n - v_n)\varphi'(u_n - v_n)\, dx\, ds\\
&\quad \le cn^{-(2-\sigma)}\Big[\iint_{\{n<|u|<2n\}} |\nabla u|^p\, dx\, ds + \iint_{\{n<|u|<2n\}} |\nabla u|\ell\, dx\, ds\\
&\quad + \iint_{\{n<|u|<2n,\,|v|<2n\}} |\nabla u|^{p-1}|\nabla v|\, dx\, ds + \iint_{\{n<|v|<2n\}} |\nabla v|\ell\, dx\, ds\Big],
\end{aligned} \tag{4.8}$$

and thanks to (2.33) we estimate the first integral in the right hand side above, since

$$n^{-(2-\sigma)}\iint_{\{n<|u|<2n\}} |\nabla u|^p\, dx\, ds \le c\iint_{\{n<|u|<2n\}} |\nabla |u|^\gamma|^p\, dx\, ds = \omega_n\,,$$

using that $|\{(t,x)\in(0,T)\times\Omega : n<|u|<2n\}| \to 0$ as $n\to\infty$. As far as the third integral is concerned, Hölder's inequality with (p,p') implies

$$\begin{aligned}
&n^{-(2-\sigma)}\iint_{\{n<|u|<2n,\,|v|<2n\}} |\nabla u|^{p-1}|\nabla v|\, dx\, ds\\
&\le cn^{-(2-\sigma)}\left(\iint_{\{n<|u|<2n\}} |\nabla u|^p\, dx\, ds\right)^{\frac{1}{p'}}\left(\iint_{\{|v|<2n\}} |\nabla v|^p\, dx\, ds\right)^{\frac{1}{p}}\\
&\le \left(\iint_{\{n<|u|<2n\}} |\nabla |u|^\gamma|^p\, dx\, ds\right)^{\frac{1}{p'}}\left(\iint_{Q_T} |\nabla |v|^\gamma|^p\, dx\, ds\right)^{\frac{1}{p}} = \omega_n\,,
\end{aligned}$$

thanks again to (2.33). Finally, we deal with the second term in (4.8) (the fourth is treated in the same way) applying again Hölder's inequality with (p,p') and so obtaining

$$\begin{aligned}
&cn^{-(2-\sigma)}\iint_{\{n<|u|<2n\}} |\nabla u|\ell\, dx\, ds\\
&\le cn^{-(2-\sigma)}\left(\iint_{\{n<|u|<2n\}} |\nabla u|^p\, dx\, ds\right)^{\frac{1}{p}}\left(\iint_{\{n<|u|<2n\}} |\ell|^{p'}\, dx\, ds\right)^{\frac{1}{p'}}\\
&\le \left(\iint_{\{n<|u|<2n\}} |\nabla |u|^\gamma|^p\, dx\, ds\right)^{\frac{1}{p}}\left(n^{-(2-\sigma)}\iint_{Q_T} |\ell|^{p'}\, dx\, ds\right)^{\frac{1}{p'}} = \omega_n.
\end{aligned}$$

A similar argument can be done for the last term in (4.7), so that we have that $I_1 = \omega_n$.

By the definition of $\theta_n(\cdot)$ and since $|\varphi(u_n - v_n)| \le cn^{\sigma-1}$, I_2 can be estimated as

$$I_2 \le \frac{c}{n^{2-\sigma}} \left[\iint_{\{n<|u|<2n\}} |a(s,x,\nabla u)||\nabla u|\,dx\,ds + \iint_{\{n<|v|<2n\}} |a(s,x,\nabla v)||\nabla v|\,dx\,ds \right],$$

and $I_2 = \omega_n$ thanks to (2.3) and Lemma 3.4 (see (3.6)).

As far as I_3 is concerned, we have that

$$|I_3| \le cn^{\sigma-1}\Bigg[\iint_{\{n<|u|<2n\}} |\theta_n(u)H(s,x,\nabla u) - H(s,x,\nabla u_n)|\,dx\,ds + \iint_{\{n<|v|<2n\}} |\theta_n(v)H(s,x,\nabla v) - H(s,x,\nabla v_n)|\,dx\,ds \Bigg]$$

by definition of $\theta_n(\cdot)$: indeed, $\theta_n(u)H(s,x,\nabla u) - H(s,x,\nabla u_n) = 0$ if $|u| \le n$ and $|\nabla u_n| = 0$ if $|u| \ge 2n$. We only consider the first term in the inequality above since the second one can be dealt with in the same way. Thanks to the growth assumption (2.11), the desired convergence of the first term follows once we prove that

$$\begin{aligned} cn^{\sigma-1} &\iint_{\{n<|u|<2n\}} |\theta_n(u)H(s,x,\nabla u) - H(s,x,\nabla u_n)|\,dx\,ds \\ &\le c\left[\iint_{\{n<|u|<2n\}} |\nabla u|^q |u|^{\sigma-1}\,dx\,ds + \iint_{\{n<|u|<2n\}} |f||u|^{\sigma-1}\,dx\,ds \right] = \omega_n. \end{aligned}$$

An application of Hölder's inequality with indices $\left(\frac{p}{q}, \frac{p^*}{p-q}, \frac{N}{p-q}\right)$, Sobolev's embedding and Theorem 3.1 (see (3.2)) lead us to

$$\begin{aligned} \iint_{\{n<|u|<2n\}} |\nabla u|^q |u|^{\sigma-1}\,dx\,ds &\le \int_0^t \left(\int_{\{n<|u(s)|\le 2n\}} |\nabla |u(s)|^\gamma|^p\,dx \right)^{\frac{q}{p}} \left(\int_{\{n<|u(s)|\le 2n\}} |u(s)|^{\sigma-1+\frac{q}{p-q}}\,dx \right)^{\frac{p-q}{p}} ds \\ &\le c \sup_{s\in[0,t]} \left(\int_\Omega |u(s)|^\sigma\,dx \right)^{\frac{p-q}{N}} \iint_{\{n<|u|<2n\}} |\nabla |u|^\gamma|^p\,dx\,ds = \omega_n \end{aligned} \tag{4.9}$$

for the same reasons given above. As far as the integral involving the forcing term is concerned, we observe that, applying the Hölder inequality with indices (m, m') and (r, r') as in (2.31), we get

$$\iint_{\{n<|u|<2n\}} |f||u|^{\sigma-1}\,dx\,ds \le \|f\|_{L^r(0,T;L^m(\Omega))} \left[\int_0^t \left(\int_{\{n<|u(s)|\le 2n\}} |u(s)|^{\gamma\frac{m'(\sigma-1)}{\gamma}}\,dx \right)^{\frac{r'}{m'}} ds \right]^{\frac{1}{r'}}.$$

We go further invoking Theorem 3.1 with $w = |u|^\gamma$ and, in particular, the inequality in (3.2) becomes

$$\int_0^t \left(\int_{\{n<|u(s)|\le 2n\}} |u(s)|^{\gamma m}\,dx \right)^{\frac{y}{m}} ds \le c_{GN} \|u\|^{\gamma(y-p)}_{L^\infty(0,T;L^\sigma(\Omega))} \iint_{\{n<|u|<2n\}} |\nabla |u|^\gamma|^p\,dx\,ds. \tag{4.10}$$

We observe that since the couple (m, r) satisfies (2.31), we have that the pair (j, y) fulfills

$$j \ge m' \frac{\sigma-1}{\gamma} \quad \text{and} \quad y \ge r' \frac{\sigma-1}{\gamma}.$$

We thus proceed through Lebesgue spaces inclusion and we deduce

$$\begin{aligned} &\|f\|_{L^r(0,T;L^m(\Omega))} \left[\int_0^t \left(\int_{\{n<|u(s)|\le 2n\}} |u(s)|^{\gamma w}\,dx \right)^{\frac{y}{w}} ds \right]^{\frac{\sigma-1}{\gamma y}} \\ &\le c\|f\|_{L^r(0,T;L^m(\Omega))} \left(\|u\|^{\gamma(y-p)}_{L^\infty(0,T;L^\sigma(\Omega))} \iint_{\{n<|u|<2n\}} |\nabla |u|^\gamma|^p\,dx\,ds \right)^{\frac{\sigma-1}{\gamma y}} = \omega_n, \end{aligned}$$

that implies $I_3 = \omega_n$. Finally, recalling that $u_0 \le v_0$ and the definition of $\theta_n(\cdot)$, we conclude that also $I_4 \le \omega_n$. so that that $R_n = \omega_n$.

We now get into the main step of the proof. Let $E_{n,k}$ be the subset of Q_t defined by

$$E_{n,k} = \{(t,x) \in Q_t : \; z_n > k \quad \text{and} \quad |\nabla z_n| > 0\}, \tag{4.11}$$

and we set

$$\mathcal{B}_{n,1} = \left[g + |\nabla u_n|^{q-1} + |\nabla v_n|^{q-1}\right] \chi_{\{u_n > v_n\}}, \qquad \mathcal{B}_{n,2} = \left[|\nabla u_n|^2 + |\nabla v_n|^2\right]^{\frac{a(2-p)}{2(2-a)}} \chi_{\{u_n > v_n\}}, \tag{4.12}$$

where the parameter $a \le p \le 2$ has to be fixed.

An application of Hölder's inequality with indices $(\frac{2}{a}, \frac{2}{2-a})$ and the inequality in (4.4) provide with the following estimate:

$$\begin{aligned}
&\iint_{Q_t} \frac{|\nabla G_k(z_n)|^a}{(G_k(z_n)+\mu)^{\frac{a}{2}(2-\sigma)}}\,dx\,ds \\
&\le \left(\iint_{E_{n,k}} \mathcal{B}_{n,2}\,dx\,ds\right)^{\frac{2-a}{2}} \left(\iint_{E_{n,k}} \frac{|\nabla G_k(z_n)|^2}{(G_k(z_n)+\mu)^{2-\sigma}}\left[|\nabla u_n|^2+|\nabla v_n|^2\right]^{\frac{p-2}{2}}\,dx\,ds\right)^{\frac{a}{2}} \\
&\le c\left(\iint_{E_{n,k}} \mathcal{B}_{n,2}\,dx\,ds\right)^{\frac{2-a}{2}} \left(\iint_{Q_t} |\nabla G_k(z_n)|\,\mathcal{B}_{n,1}\left[(G_k(z_n)+\mu)^{\sigma-1}-\mu^{\sigma-1}\right]dx\,ds+\omega_n\right)^{\frac{a}{2}}.
\end{aligned} \tag{4.13}$$

We recall (2.30) (i.e. we know that $g \in L^d(\Omega)$ for $d = \frac{N(q-(p-1))-p+2q}{q-1}$) and we set a such that $\frac{a(2-p)}{2-a} = d(q-1) = N(q-(p-1))+2q-p$, namely

$$a = \frac{2d(q-1)}{2-p+d(q-1)} = \frac{2\left(N(q-(p-1))-p+2q\right)}{(N+2)(q-(p-1))}.$$

Then, the gradient regularity (3.5) contained in Lemma 3.4 applied on both $|\nabla u_n|$ and $|\nabla v_n|$ (we recall also (2.33)) implies that the first integral in the right hand side of (4.13) is finite.

Now, let us focus on the second one. Hölder's inequality with indices $\left(a, d, \frac{2}{\sigma}\frac{p(N\gamma+\sigma)}{N}\right)$ (indeed, $1-\frac{1}{a}-\frac{1}{d} = \frac{1}{2}\frac{\sigma(p-q)}{\sigma(p-q)-p+2q} = \frac{2}{\sigma}\frac{p(N\gamma+\sigma)}{N}$ by definitions of σ and γ) yields to

$$\begin{aligned}
&\iint_{Q_t} |\nabla G_k(z_n)|\,\mathcal{B}_{n,1}\left[(G_k(z_n)+\mu)^{\sigma-1}-\mu^{\sigma-1}\right]dx\,ds \\
&\le \iint_{Q_t}\left[|\nabla G_k(z_n)|\,(G_k(z_n)+\mu)^{\frac{\sigma-2}{2}}\,\mathcal{B}_{n,1}\left[(G_k(z_n)+\mu)^{\sigma-1}-\mu^{\sigma-1}\right]^{\frac12}(G_k(z_n)+\mu)^{\frac12}\right]dx\,ds \\
&\le \left(\iint_{Q_t} \frac{|\nabla G_k(z_n)|^a}{(G_k(z_n)+\mu)^{\frac{a}{2}(2-\sigma)}}\,dx\,ds\right)^{\frac1a}\left(\iint_{E_{n,k}} \mathcal{B}_{n,1}^d\,dx\,ds\right)^{\frac1d}\times \\
&\times\left(\iint_{Q_t}\left(\left[(G_k(z_n)+\mu)^{\sigma-1}-\mu^{\sigma-1}\right]^{\frac12}(G_k(z_n)+\mu)^{\frac12}\right)^{\frac{2p(N\gamma+\sigma)}{N\sigma}}dx\,ds\right)^{\frac{N\sigma}{2p(N\gamma+\sigma)}}
\end{aligned} \tag{4.14}$$

and then we take advantage of (4.14) in (4.13), we obtain

$$\begin{aligned}
&\iint_{Q_t} \frac{|\nabla G_k(z_n)|^a}{(G_k(z_n)+\mu)^{\frac{a}{2}(2-\sigma)}}\,dx\,ds \\
&\le c\left(\iint_{E_{n,k}} \mathcal{B}_{n,1}^d\,dx\,ds\right)^{\frac{a}{d}}\left(\iint_{E_{n,k}} \mathcal{B}_{n,2}\,dx\,ds\right)^{2-a}\times \\
&\times\left(\iint_{Q_t}\left(\left[(G_k(z_n)+\mu)^{\sigma-1}-\mu^{\sigma-1}\right]^{\frac12}(G_k(z_n)+\mu)^{\frac12}\right)^{\frac{2p(N\gamma+\sigma)}{N\sigma}}dx\,ds\right)^{\frac{N\sigma}{2p(N\gamma+\sigma)}a}+\omega_n.
\end{aligned} \tag{4.15}$$

We observe that the first two integrals in the right hand side above are bounded thanks to (2.30), the definition of a and Lemma 3.4. Moreover, using (4.15) in (4.14) leads to

$$\begin{aligned}
&\iint_{Q_t} |\nabla G_k(z_n)|\,\mathcal{B}_{n,1}\left[(G_k(z_n)+\mu)^{\sigma-1}-\mu^{\sigma-1}\right]dx\,ds \\
&\le c\left(\iint_{E_{n,k}} \mathcal{B}_{n,1}^d\,dx\,ds\right)^{\frac{2}{d}}\left(\iint_{E_{n,k}} \mathcal{B}_{n,2}\,dx\,ds\right)^{\frac{2-a}{a}}\times \\
&\times\left(\iint_{Q_t}\left(\left[(G_k(z_n)+\mu)^{\sigma-1}-\mu^{\sigma-1}\right]^{\frac12}(G_k(z_n)+\mu)^{\frac12}\right)^{\frac{2p(N\gamma+\sigma)}{N\sigma}}dx\,ds\right)^{\frac{N\sigma}{p(N\gamma+\sigma)}}+\omega_n
\end{aligned} \tag{4.16}$$

and the uniform estimate of the right hand side in (4.4) is closed.

Next, we observe that the definitions of a, γ and σ imply that

$$\frac{2}{\sigma}\frac{p(N\gamma+\sigma)}{N} = a\frac{N+2}{N}$$

where $a\frac{N+2}{N}$ is the Gagliardo-Nirenberg regularity exponent applied with spaces

$$L^\infty(0,T;L^2(\Omega))\cap L^a(0,T;W_0^{1,a}(\Omega)).$$

In particular, the inequality in (3.2) applied to the the function

$$\left[(G_k(z_n)+\mu)^{\sigma-1}-\mu^{\sigma-1}\right]^{\frac12}(G_k(z_n)+\mu)^{\frac12}$$

gives

$$\begin{aligned}&\iint_{Q_t}\left(\left[(G_k(z_n)+\mu)^{\sigma-1}-\mu^{\sigma-1}\right]^{\frac12}(G_k(z_n)+\mu)^{\frac12}\right)^{a\frac{N+2}{N}}dx\,ds\\ &\le c_{GN}\,\|G_k(z_n)+\mu\|_{L^\infty(0,t;L^\sigma(\Omega))}^{\frac{a\sigma}{N}}\iint_{Q_t}\frac{|\nabla(G_k(z_n))|^a}{(G_k(z_n)+\mu)^{\frac a2(2-\sigma)}}\,dx\,ds.\end{aligned}\tag{4.17}$$

So far, we know that the second integral in (4.17) is bounded thanks to (4.15). Furthermore, it holds from (4.4) and (4.16) that

$$\begin{aligned}\|G_k(z_n)+\mu\|_{L^\infty(0,t;L^\sigma(\Omega))}^{\sigma}&\le\int_\Omega\Phi_k(z_n(t))\,dx\\ &\le c\left(\iint_{E_{n,k}}\mathcal{B}_{n,1}^d\,dx\,ds\right)^{\frac2d}\left(\iint_{E_{n,k}}\mathcal{B}_{n,2}\,dx\,ds\right)^{\frac{2-a}{a}}\times\\ \times\left(\iint_{Q_t}\left(\left[(G_k(z_n)+\mu)^{\sigma-1}-\mu^{\sigma-1}\right]^{\frac12}(G_k(z_n)+\mu)^{\frac12}\right)^{a\frac{N+2}{N}}dx\,ds\right)^{\frac{2N}{a(N+2)}}&+\omega_n\end{aligned}$$

and then, using (4.15) and (4.17), it follows that

$$\begin{aligned}&\iint_{Q_t}\left(\left[(G_k(z_n)+\mu)^{\sigma-1}-\mu^{\sigma-1}\right]^{\frac12}(G_k(z_n)+\mu)^{\frac12}\right)^{a\frac{N+2}{N}}dx\,ds\\ &\le c\left(\iint_{E_{n,k}}\mathcal{B}_{n,1}^d\,dx\,ds\right)^{\frac{2a}{dN}}\left(\iint_{E_{n,k}}\mathcal{B}_{n,2}\,dx\,ds\right)^{\frac{2-a}{N}}\times\\ &\times\left(\iint_{Q_t}\left(\left[(G_k(z_n)+\mu)^{\sigma-1}-\mu^{\sigma-1}\right]^{\frac12}(G_k(z_n)+\mu)^{\frac12}\right)^{a\frac{N+2}{N}}dx\,ds\right)+\omega_n^{\frac aN}.\end{aligned}$$

Finally, passing to the limit first with respect to n (we apply Lebesgue Theorem on the right hand side) and subsequently as $\mu\to0$, we obtain:

$$\iint_{Q_t}G_k(u-v)^{\frac{a\sigma}{2}\frac{N+2}{N}}dx\,ds\le c\left(\iint_{E_k}\mathcal{B}_1^d\,dx\,ds\right)^{\frac{2a}{dN}}\left(\iint_{E_k}\mathcal{B}_2\,dx\,ds\right)^{\frac{2-a}{N}}\left(\iint_{Q_t}G_k(u-v)^{\frac{a\sigma}{2}\frac{N+2}{N}}dx\,ds\right)$$

where

$$E_k=\{(t,x)\in Q_t:\quad u-v>k\text{ and }|\nabla(u-v)|>0\}\tag{4.18}$$

while

$$\mathcal{B}_1=\left[g+|\nabla u|^{q-1}+|\nabla v|^{q-1}\right]\chi_{\{u>v\}}\qquad\text{and}\qquad\mathcal{B}_2=\left[|\nabla u|^2+|\nabla v|^2\right]^{\frac{N(q-(p-1))+2q-p}{2}}\chi_{\{u>v\}}.$$

We conclude the proof applying Lemma 3.2 with $\rho=a$, $\nu=2$ to the function $G_k(u-v)$. □

Using the same ideas, adapted to the case of L^1 data and low values of q, we prove now Theorem 2.11 and Theorem 2.7.

Proof of Theorem 2.11 and Theorem 2.7. Before getting into the real proof, we recall (2.35) and observe that

$$[g+|\nabla u|^{q-1}+|\nabla v|^{q-1}]\chi_{\{u>v\}}\in L^d(Q_T).\tag{4.19}$$

Indeed, Lemma 3.5 provides us that $|\nabla u_+|, |\nabla v_-|\in L^q(Q_T)$ for every $q<2-\frac{N}{N+1}$. So $|\nabla u_+|^{q-1}$ and $|\nabla v_-|^{q-1}$ belong to $L^r(Q_T)$ for every $r>N+2$. Furthermore, being $(u-v)_+$ a subsolution itself and reasoning as in the just mentioned Lemma with $T_k((z_n)_+)$, then $(u-v)_+$ inherits the $L^q(0,T;W^{1,q}(\Omega))$ regularity for every $q<2-\frac{N}{N+1}$. We underline that we use the fact that ω_n is assumed to be uniformly bounded in n - it will soon proved - and also converging to 0. Then, the regularity in (4.19) follows since

$$|\nabla v|\chi_{\{u>v\}}\le|\nabla v|\chi_{\{v<0\}}+|\nabla v|\chi_{\{0<v<u\}}\le|\nabla v_-|+|\nabla u_+|+|\nabla(u-v)_+|,$$

and

$$|\nabla u|\chi_{\{u>v\}}\le|\nabla v_-|+|\nabla u_+|+|\nabla(u-v)_+|.$$

We use the same notation of Theorem 2.10 (see (4.1) and (4.2)); in particular wee take again $S(u) = S_n(u)$ in (4.3) and $S(v) = S_n(v)$ in the formulation of the supersolution, and we set

$$\varphi = \varphi(z_n) = \frac{1}{\mu^\lambda} - \frac{1}{(G_k(z_n)+\mu)^\lambda} \quad \text{with} \quad \lambda = \frac{N+1}{N}(p-q) - 1 \quad \text{and} \quad \mu > 0$$

getting

$$\begin{aligned}\int_\Omega \Phi_k(z_n(t))\,dx + \lambda\alpha \iint_{Q_t} \frac{|\nabla(G_k(z_n))|^2}{(G_k(z_n)+\mu)^{\lambda+1}} \left[|\nabla u_n|^2 + |\nabla v_n|^2\right]^{\frac{p-2}{2}} dx\,ds \\ \le \frac{c_1}{\mu^\lambda} \iint_{Q_t} |\nabla G_k(z_n)|\,\mathcal{B}_{n,1}\,dx\,ds + \frac{R_n}{\mu^\lambda}\end{aligned} \tag{4.20}$$

where $\mathcal{B}_{n,1}$ and R_n have been defined in (4.12) and (4.6), respectively.

Once again, we have that $\lim_{n\to\infty} R_n = 0$; the proof of this fact is quite similar to the one contained in Theorem 2.10. We just recall the decompositions (4.5)–(4.6) and that, using the current choice of $\varphi(\cdot)$ and the asymptotic energy condition (2.7), we deduce that $I_1 + I_2 = \omega_n$. The terms I_3 and I_4 follow as in Theorem 2.10, just observing that now f only belong to $L^1(Q_T)$, using again (2.7).

The uniform boundedness of the right hand side in (4.20) follows from the above remark by (4.19).

Now, let the parameter $a \le p$ be such that $\frac{a(2-p)}{2-a} = q$, i.e. $a = \frac{2q}{q-p+2}$. Then, recalling the inequality in (4.20), we obtain

$$\iint_{Q_t} \frac{|\nabla G_k(z_n)|^a}{(G_k(z_n)+\mu)^{a\frac{\lambda+1}{2}}}\,dx\,ds \le \frac{c}{\mu^{\frac{\lambda a}{2}}} \left(\iint_{E_{n,k}} |\nabla G_k(z_n)|\,\mathcal{B}_{n,1}\,dx\,ds + \omega_n\right)^{\frac{a}{2}} \left(\iint_{E_{n,k}} \mathcal{B}_{n,2}\,dx\,ds\right)^{\frac{2-a}{2}} \tag{4.21}$$

for $\mathcal{B}_{n,1}$, $\mathcal{B}_{n,2}$ as in (4.12), $E_{n,k}$ as in (4.11). In particular, (4.20) provides us with

$$\int_\Omega \Phi_k(z_n(t))\,dx \le \frac{\bar\gamma}{\mu^\lambda} \iint_{E_{n,k}} |\nabla G_k(z_n)|\,\mathcal{B}_{n,1}\,dx\,ds + \frac{\omega_n}{\mu^\lambda}. \tag{4.22}$$

Furthermore, the definition of $\Phi_k(\cdot)$ implies

$$\Phi_k(w) \ge \frac{c}{\mu^\lambda} G_k(w) + c, \tag{4.23}$$

with c independent from μ, so that

$$\int_\Omega \Phi_k(z_n(t))\,dx \ge \frac{c}{\mu^\lambda} \int_\Omega G_k(z_n(t))\,dx + c.$$

We use (4.23) in (4.22), so that

$$\int_\Omega G_k(z_n(t))\,dx \le c \iint_{E_{n,k}} |\nabla G_k(z_n)|\,\mathcal{B}_{n,1}\,dx\,ds + \omega_n + c\mu^\lambda$$

that becomes, taking the limits as $\mu \to 0$, and then as $n \to +\infty$,

$$\int_\Omega G_k(u(t) - v(t))\,dx \le c \iint_{E_{n,k}} |\nabla G_k(u-v)|\,\mathcal{B}_1\,dx\,ds,$$

where the last convergence follows thanks to Lemma 3.5.

Furthermore, letting $n \to \infty$ also in (4.21), we get

$$\iint_{Q_t} \frac{|\nabla G_k(u-v)|^a}{[G_k(u-v)+\mu]^{a\frac{\lambda+1}{2}}}\,dx\,ds \le \frac{c}{\mu^{\frac{\lambda a}{2}}} \left(\iint_{Q_t} |\nabla G_k(u-v)|\,\mathcal{B}_1\,dx\,ds\right)^{\frac{a}{2}} \left(\iint_{E_k} \mathcal{B}_2\,dx\,ds\right)^{\frac{2-a}{2}}$$

with E_k as in (4.18) and

$$\mathcal{B}_1 = \left[g + |\nabla u|^{q-1} + |\nabla v|^{q-1}\right] \chi_{\{u>v\}}, \qquad \mathcal{B}_2 = \left[|\nabla u|^2 + |\nabla v|^2\right]^{\frac{q}{2}} \chi_{\{u>v\}}.$$

We now apply Lemma 3.6 with

$$\rho = a, \quad \nu = \frac{\lambda a}{2}, \quad m = 1, \quad \gamma = a\frac{1+\lambda}{2}$$

and

$$M = c\left(\iint_{Q_t} |\nabla G_k(u-v)|\,\mathcal{B}_1\,dx\,ds\right)^{\frac{a}{2}} \left(\iint_{E_k} \mathcal{B}_2\,dx\,ds\right)^{\frac{2-a}{2}}, \qquad L = c\iint_{Q_t} |\nabla G_k(u-v)|\,\mathcal{B}_1\,dx\,ds.$$

In particular, the estimate in (3.9) holds with

$$b = a\frac{N(1-\lambda)+2}{2(N+1)} = q$$

by the definitions of a, λ and

$$\|\nabla G_k(u-v)\|_{L^q(Q_T)} \le c\left(\iint_{Q_t} |\nabla G_k(u-v)|\,\mathcal{B}_1\,dx\,ds\right)\left(\iint_{E_k} \mathcal{B}_2\,dx\,ds\right)^{\frac{(2-a)(N+1)}{a(N+2)}}.$$

We apply Hölder's inequality with (q,q') obtaining

$$\|\nabla G_k(u-v)\|_{L^q(Q_T)} \le c\|\nabla G_k(u-v)\|_{L^q(Q_T)}\left(\iint_{Q_t} \mathcal{B}_1^{\frac{q}{q-1}}\,dx\,ds\right)^{\frac{q-1}{q}}\left(\iint_{E_k} \mathcal{B}_2\,dx\,ds\right)^{\frac{(2-a)(N+1)}{a(N+2)}}.$$

Observe that, since $\frac{q}{q-1} \searrow \frac{p(N+1)-N}{p(N+1)-(2N+1)}$ as $q \nearrow p - \frac{N}{N+1}$, the integral involving $\mathcal{B}_1$ is bounded (up to choose q closer to the threshold). Then we conclude by applying Lemma 3.2 with $\rho = q$. □

5. Proofs in the case $2 \le p < N$

We start by proving the results for $p = 2$, since their proofs are different from those of the case $p > 2$.

5.1. The case $p = 2$.

Proof of Theorem 2.8. We follow the same notation that we have used for the proof of Theorem 2.10, by defining u_n, v_n as in (4.1). Thus we consider the inequalities in (2.6) satisfied by the sub/supersolutions with $S(u) = u_n$ and $S(v) = v_n$ respectively so that, we have

$$\int_\Omega u_n(t)\varphi(t)\,dx + \iint_{Q_t} A(s,x)\nabla u_n\cdot\nabla\varphi + A(s,x)\nabla u\cdot\nabla u\,\theta_n'(u)\varphi\,dx\,ds \le \iint_{Q_t} H(t,x,\nabla u)\theta_n(u)\varphi\,dx\,ds + \int_\Omega u_n(0)\varphi(0)\,dx$$

and

$$\int_\Omega (1-\varepsilon)v_n(t)\varphi(t)\,dx + \iint_{Q_t} A(s,x)\nabla((1-\varepsilon)v_n)\cdot\nabla\varphi + A(s,x)\nabla((1-\varepsilon)v)\cdot\nabla v\,\theta_n'(v)\varphi\,dx\,ds \ge (1-\varepsilon)\iint_{Q_t} H(t,x,\nabla v)\theta_n(v)\varphi\,dx\,ds + (1-\varepsilon)\int_\Omega v_n(0)\varphi(0)\,dx$$

where the inequality related to the supersolution has been multiplied by $(1-\varepsilon)$, for $\varepsilon \in (0,1)$. Then, taking into account the difference between the inequalities above, we get

$$\begin{aligned}\int_\Omega [u_n(t)-(1-\varepsilon)v_n(t)]\,\varphi(t)\,dx &+ \iint_{Q_t} A(s,x)\nabla\left(u_n-(1-\varepsilon)v_n\right)\cdot\nabla\varphi\,dx\,ds\\ &+ \iint_{Q_t}\left[A(s,x)\nabla u\cdot\nabla u\,\theta_n'(u) - A(s,x)\nabla((1-\varepsilon)v)\cdot\nabla v\,\theta_n'(v)\right]\varphi\,dx\,ds\\ \le \iint_{Q_t}&\left[H(t,x,\nabla u)\theta_n(u)-(1-\varepsilon)H(t,x,\nabla v)\theta_n(v)\right]\varphi\,dx\,ds + \int_\Omega z_n^\varepsilon(0)\varphi(0)\,dx.\end{aligned}$$

We use the hypothesis (2.23) in order to we rewrite the above inequality as

$$\begin{aligned}\int_\Omega (u_n(t)-(1-\varepsilon)v_n(t))\,\varphi(t)\,dx &+ \iint_{Q_t} A(s,x)\nabla\left(u_n-(1-\varepsilon)v_n\right)\cdot\nabla\varphi\,dx\,ds\\ \le \iint_{Q_t} [H_1(s,x,\nabla u_n)-(1-\varepsilon)H_1(s,x,\nabla v_n)]\,\varphi &+ [H_2(s,x,\nabla u_n)-(1-\varepsilon)H_2(s,x,\nabla v_n)]\,\varphi\,dx\,ds + R_n\,,\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}R_n = &\iint_{Q_t}\left[A(s,x)\nabla u\cdot\nabla u\,\theta_n'(u) - A(s,x)\nabla((1-\varepsilon)v)\cdot\nabla v\,\theta_n'(v)\right]\varphi\,dx\,ds\\ &+ \iint_{Q_t}\left[H(s,x,\nabla u)\theta_n(u)-(1-\varepsilon)H(s,x,\nabla v)\theta_n(v) - H(s,x,\nabla u_n) + (1-\varepsilon)H(s,x,\nabla v_n)\right]\varphi\,dx\,ds\\ &+ \int_\Omega z_n^\varepsilon(0)\varphi(0)\,dx.\end{aligned}$$

Let us observe that the conditions (2.25) and (2.26) imply that

$$
\begin{aligned}
&H_2(s,x,\nabla u_n)-(1-\varepsilon)H_2(s,x,\nabla v_n)\\
&\leq |H_2(s,x,\nabla u_n)-H_2(s,x,(1-\varepsilon)\nabla v_n)|+H_2(s,x,(1-\varepsilon)\nabla v_n)-(1-\varepsilon)H_2(s,x,\nabla v_n)\\
&\leq c_2|\nabla\left(u_n-(1-\varepsilon)v_n\right)|\leq c\varepsilon\left[\frac{|\nabla\left(u_n-(1-\varepsilon)v_n\right)|^q}{\varepsilon^q}+1\right],
\end{aligned}
$$

byYoung's inequality. On the other hand, since $H_1(t,x,\xi)$ satisfies (2.24) (i.e. the convexity assumption with respect to ξ), we have

$$
H_1(s,x,\nabla u_n)-(1-\varepsilon)H_1(s,x,\nabla v_n)\leq \varepsilon H_1\left(t,x,\frac{\nabla\left(u_n-(1-\varepsilon)v_n\right)}{\varepsilon}\right).
$$

Finally, the growth assumption on $H_1(s,x,\xi)$ contained in (2.11) allows us to improve (5.1) as

$$
\begin{aligned}
&\int_\Omega \left(u_n(t)-(1-\varepsilon)v_n(t)\right)\varphi(t)\,dx+\iint_{Q_t}A(s,x)\nabla\left(u_n-(1-\varepsilon)v_n\right)\cdot\nabla\varphi\,dx\,ds\\
&\qquad\leq \varepsilon\iint_{Q_t}\left(c_1\frac{|\nabla\left(u_n-(1-\varepsilon)v_n\right)|^q}{\varepsilon^q}+\tilde f\right)dx\,ds+R_n
\end{aligned}
$$

where $\tilde f=f+c$. In particular, the inequality above can be written in terms of the function

$$
z_n^\varepsilon(t,x)=\frac{e^{-t}}{\varepsilon}\big(u_n(t,x)-(1-\varepsilon)v_n(t,x)\big) \tag{5.2}
$$

as

$$
\begin{aligned}
&\int_\Omega z_n^\varepsilon(t)\varphi(t)\,dx+\iint_{Q_t}z_n^\varepsilon\varphi\,dx\,ds+\iint_{Q_t}A(s,x)\nabla z_n^\varepsilon\cdot\nabla\varphi\,dx\,ds\\
&\qquad\leq\iint_{Q_t}\Big(c_1|\nabla z_n^\varepsilon|^q+\tilde f\Big)\varphi\,dx\,ds+\frac{R_n}{\varepsilon}.
\end{aligned} \tag{5.3}
$$

We consider the inequality in (5.3) with

$$
\varphi(z_n^\varepsilon)=\int_0^{G_k(z_n^\varepsilon)}(\mu+|w|)^{\sigma-2}\,dw,\qquad \mu>0,
$$

(again, we recall the density results in [PPP, Proposition 4.2]) getting

$$
\begin{aligned}
&\int_\Omega \Phi_k(z_n^\varepsilon(t))\,dx+\iint_{Q_t}z_n^\varepsilon\varphi\,dx\,ds+\alpha\iint_{Q_t}|\nabla\Psi_k(G_k(z_n^\varepsilon))|^2\,dx\,ds\\
&\leq c_1\iint_{Q_t}|\nabla G_k(z_n^\varepsilon)|^q\bigg(\int_0^{G_k(z_n^\varepsilon)}(\mu+|w|)^{\sigma-2}\,dz\bigg)dx\,ds\\
&+\iint_{Q_t}|\tilde f|\chi_{\{|\tilde f|>k\}}\bigg(\int_0^{G_k(z_n^\varepsilon)}(\mu+|w|)^{\sigma-2}\,dz\bigg)dx\,ds+\iint_{Q_t}|\tilde f|\chi_{\{|\tilde f|\leq k\}}\varphi\,dx\,ds+\frac{R_n}{\varepsilon},
\end{aligned}
$$

with

$$
\Psi_\mu(G_k(z_n^\varepsilon))=\int_0^{G_k(z_n^\varepsilon)}(\mu+|w|)^{\frac{\sigma-2}{2}}\,dw\qquad\text{and}\qquad \Phi_k(z_n^\varepsilon(t)))=\int_0^{G_k(z_n^\varepsilon(t))}\varphi(w)\,dw.
$$

Observe that since φ is supported where $z_n^\varepsilon\geq k$, then

$$
\iint_{Q_t}z_n^\varepsilon\varphi\,dx\,ds-\iint_{Q_t}|\tilde f|\chi_{\{|\tilde f|\leq k\}}\varphi\,dx\,ds\geq k\iint_{Q_t}\varphi\,dx\,ds-k\iint_{Q_t}\varphi\,dx\,ds\geq 0\,,
$$

and we are reduced to study

$$
\begin{aligned}
&\int_\Omega \Phi_k(z_n^\varepsilon(t))\,dx+\alpha\iint_{Q_t}|\nabla\Psi_k(G_k(z_n^\varepsilon))|^2\,dx\,ds\\
&\leq c_1\iint_{Q_t}|\nabla G_k(z_n^\varepsilon)|^q\bigg(\int_0^{G_k(z_n^\varepsilon)}(\mu+|w|)^{\sigma-2}\,dw\bigg)dx\,ds+\iint_{Q_t}|\tilde f|\chi_{\{|\tilde f|>k\}}\bigg(\int_0^{G_k(z_n^\varepsilon)}(\mu+|w|)^{\sigma-2}\,dw\bigg)dx\,ds+\frac{R_n}{\varepsilon}.
\end{aligned}
$$

We just note that, by definitions of $\theta_n(\cdot)$, $\varphi(\cdot)$ and thanks to (2.22), (2.33), the proof of $\lim_{n\to\infty}R_n=0$ follows reasoning as in Theorem 2.10.

We observe that the definition of $\Psi_k(\cdot)$ combined with Hölder's inequality with indices $\left(\frac{1}{2-q}, \frac{1}{q-1}\right)$ and also an application Young inequality with $\left(\frac{2}{q}, \frac{2}{2-q}\right)$ leads to

$$
\begin{aligned}
&\iint_{Q_t} |\nabla G_k(z_n^\varepsilon)|^q \left(\int_0^{G_k(z_n^\varepsilon)} (\mu+|w|)^{\sigma-2}\,dw\right) dx\,ds\\
&\le c \iint_{Q_t} |\nabla \Psi_k(G_k(z_n^\varepsilon))|^q \left(\int_0^{G_k(z_n^\varepsilon)} (\mu+|w|)^{(\sigma-2)\frac{2-q}{2}}\,dw\right) dx\,ds\\
&\le c \iint_{Q_t} |\nabla \Psi_k(G_k(z_n^\varepsilon))|^q |\Psi_k(G_k(z_n^\varepsilon))|^{2-q} G_k(z_n^\varepsilon)^{q-1}\,dx\,ds\\
&\le \frac{\alpha}{2} \iint_{Q_t} |\nabla \Psi_k(G_k(z_n^\varepsilon))|^2\,dx\;ds + c\iint_{Q_t} |\Psi_k(G_k(z_n^\varepsilon))|^2 G_k(z_n^\varepsilon)^{\frac{2(q-1)}{2-q}}\,dx\,ds.
\end{aligned}
$$

We now focus on the term involving the source f. We recall the estimate in Theorem 2.10 and apply Hölder's inequality with (m,m') and (r,r') getting

$$
\begin{aligned}
&\iint_{Q_t} |\tilde f|\chi_{\{|\tilde f|>k\}} \left(\int_0^{G_k(z_n^\varepsilon)} (\mu+|w|)^{\sigma-2}\,dw\right) dx\,ds\\
&\le c \iint_{Q_t} |\tilde f|\chi_{\{|\tilde f|>k\}} \left(\int_0^{G_k(z_n^\varepsilon)} (\mu+|w|)^{\frac{\sigma-2}{2}}\,dw\right)^{\frac{2}{\sigma'}} dx\,ds\\
&\le c\| |\tilde f|\chi_{\{|\tilde f|>k\}}\|_{L^r(0,T;L^m(\Omega))}\, \|\Psi_k(G_k(z_n^\varepsilon))\|^{2\frac{\sigma-1}{\sigma}}_{L^{2r'\frac{\sigma-1}{\sigma}}(0,t;L^{2m'\frac{\sigma-1}{\sigma}}(\Omega))}
\end{aligned}
$$

where r, m verifies (2.31). Finally, we gather together the estimates above and find that

$$
\begin{aligned}
\int_\Omega \Phi_k(z_n^\varepsilon(t))\,dx + \frac{\alpha}{2}\iint_{Q_t} |\nabla\Psi_k(G_k(z_n^\varepsilon))|^2\,dx\,ds &\le c\iint_{Q_t} |\Psi_k(G_k(z_n^\varepsilon))|^2 G_k(z_n^\varepsilon)^{\frac{2(q-1)}{2-q}}\,dx\,ds\\
+c\| |\tilde f|\chi_{\{|\tilde f|>k\}}\|_{L^r(0,T;L^m(\Omega))} \|\Psi_k(G_k(z_n^\varepsilon))\|^{2\frac{\sigma-1}{\sigma}}_{L^{2r'\frac{\sigma-1}{\sigma}}(0,t;L^{2m'\frac{\sigma-1}{\sigma}}(\Omega))} &+ \omega_n.
\end{aligned}
\tag{5.4}
$$

Then, since $\sigma + 2\frac{q-1}{2-q} = 2\frac{N+\sigma}{N}$ (indeed, $\Psi_k(z_n^\varepsilon) \le \frac{c}{\varepsilon}(|u|^{\frac{\sigma}{2}} + |u|^{\frac{\sigma}{2}})$) and thanks to (2.33), the energy integral in the left hand side is uniformly bounded.

Moreover by Theorem 3.1 we have that

$$
\iint_{Q_t} |\Psi_k(G_k(z_n^\varepsilon))|^2 G_k(z_n^\varepsilon)^{\frac{2(q-1)}{2-q}}\,dx\,ds \le c\|G_k(z_n^\varepsilon)\|^{q-1}_{L^\infty(0,t;L^\sigma(\Omega))} \int_0^t \|\nabla\Psi_k(G_k(z_n^\varepsilon(s)))\|^2_{L^2(\Omega)}\,ds
$$

and, reasoning as in (4.10), we get

$$
\|\Psi_k(G_k(z_n^\varepsilon))\|^{2\frac{\sigma-1}{\sigma}}_{L^{2r'\frac{\sigma-1}{\sigma}}(0,t;L^{2m'\frac{\sigma-1}{\sigma}}(\Omega))} \le c\|G_k(z_n^\varepsilon)\|^{\frac{\sigma}{2}(y-2)}_{L^\infty(0,t;L^\sigma(\Omega))} \int_0^t \|\nabla\Psi_k(G_k(z_n^\varepsilon(s)))\|^2_{L^2(\Omega)}\,ds,
$$

so that the inequality (5.4) reads as

$$
\begin{aligned}
&\int_\Omega \Phi_k(z_n^\varepsilon(t))\,dx + \frac{\alpha}{2}\iint_{Q_t} |\nabla\Psi_\mu(G_k(z_n^\varepsilon))|^2\,dx\,ds\\
&\le C_1\left[\|G_k(z_n^\varepsilon)\|^{q-1}_{L^\infty(0,t;L^\sigma(\Omega))} + \|G_k(z_n^\varepsilon)\|^{\frac{\sigma}{2}(y-2)}_{L^\infty(0,t;L^\sigma(\Omega))}\right] \int_0^t \|\nabla\Psi_k(G_k(z_n^\varepsilon(s)))\|^2_{L^2(\Omega)}\,ds\\
&\quad + C_2\| |\tilde f|\chi_{\{|\tilde f|>k\}}\|^{\frac{y\sigma}{y\sigma-2(\sigma-1)}}_{L^r(0,T;L^m(\Omega))} + \int_\Omega \Phi_k(v_0)\,dx + \omega_n
\end{aligned}
$$

for any $\mu > 0$, where we have used that $z_n^\varepsilon(0) \le v_0$.
Now, reasoning as in the proof of the a priori estimates contained in [Ma]. We fix a value δ_0 such that $\max\left\{\delta_0^{\frac{q-1}{\sigma}}, \delta_0^{\frac{y-2}{2}}\right\} = \frac{1}{2C_1}\frac{\alpha}{2}$ and we take $k \ge k_0$ such that

$$
\int_\Omega G_k(v_0)^\sigma\,dx + C_2\| |\tilde f|\chi_{\{|\tilde f|>k\}}\|^{\frac{y\sigma}{y\sigma-2(\sigma-1)}}_{L^r(0,T;L^m(\Omega))} < \delta_0
$$

for any $k \ge k_0$. We also define

$$
T^* = \sup\left\{\tau > 0 : \quad \|G_k(z_n^\varepsilon(s))\|^\sigma_{L^\sigma(\Omega)} \le \delta_0 \quad \forall s \le \tau\right\}
$$

which is strictly positive by (2.13) and since $u_0 \le v_0$ in Ω. Note also that T^* continuously depends on n by (2.13).

Then, for $k \geq k_0$ and for any $t \leq T^*$ we have

$$\sup_{t\in[0,T^*]} \int_\Omega \Phi_k(z_n^\varepsilon(t))\,dx < \delta_0$$

and, letting $\mu \to 0$, we deduce

$$\sup_{t\in[0,T^*]} \int_\Omega G_k(z_n^\varepsilon(t))^\sigma\,dx < \delta_0. \tag{5.5}$$

Now, if $T^* < T$, then (5.5) would be in contrast with both the continuity regularity in (2.13) and the definition of T^*, so (5.5) holds up to T.
We thus deduce a bound, uniform in ε, for the function ${z_n}^\varepsilon(t)$ in $L^\sigma(\Omega)$. Indeed, we have

$$\sup_{t\in[0,T]} \int_\Omega z_n^\varepsilon(t)^\sigma\,dx \leq \delta_0 + k_0^\sigma T|\Omega|$$

and then, letting $n \to \infty$ and recalling the definition of z_n in (5.2), leads to

$$\int_\Omega [u(t) - (1-\varepsilon)v(t)]^\sigma\,dx \leq \varepsilon^\sigma c$$

which, letting $\varepsilon \to 0$, implies $u \leq v$ in Q_T and thus the assertion is proved. □

Next we prove Theorem 2.9.

Proof of Theorem 2.9. We start recalling the inequality in (5.3), with z_n^ε defined in (5.2), and we set

$$\varphi(z_n^\varepsilon) = 1 - \frac{1}{(1+G_k(z_n^\varepsilon))^\mu}, \qquad \text{with} \qquad \mu = \frac{N+1}{N}(2-q) - 1 < 1,$$

so we get

$$\begin{aligned}\int_\Omega \Phi(z_n^\varepsilon(t))\,dx + \mu\alpha \iint_{Q_t} \frac{|\nabla G_k(z_n^\varepsilon)|^2}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\,dx\,ds + \iint_{Q_t} z_n^\varepsilon \left[1 - \frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right] dx\,ds \\ \leq c_1 e^{(q-1)T} \iint_{Q_t} |\nabla G_k(z_n^\varepsilon)|^q \left[1 - \frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right] dx\,ds + \iint_{Q_t} |\tilde f|\chi_{\{|f|>k\}} \left[1 - \frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right] dx\,ds \\ + \iint_{Q_t} |\tilde f|\chi_{\{|f|\leq k\}} \left[1 - \frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right] dx\,ds + \int_\Omega \Phi(z_n^\varepsilon(0))\,dx + R_n,\end{aligned}$$

where $\Phi(w) = \int_0^w \varphi(y)dy$. Again, we observe that

$$\iint_{Q_t} z_n^\varepsilon \left[1 - \frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right] dx\,ds - \iint_{Q_t} |\tilde f|\chi_{\{|f|\leq k\}} \left[1 - \frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right] dx\,ds \geq 0$$

so we drop it, and we just deal with

$$\begin{aligned}\int_\Omega \Phi(z_n^\varepsilon(t))\,dx + \mu\alpha \iint_{Q_t} \frac{|\nabla G_k(z_n^\varepsilon)|^2}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\,dx\,ds \\ \leq c_1 e^{(q-1)T} \iint_{Q_t} |\nabla G_k(z_n^\varepsilon)|^q \left[1 - \frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right] dx\,ds + \iint_{Q_t} |\tilde f|\chi_{\{|f|>k\}} \left[1 - \frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right] dx\,ds \\ + \int_\Omega \Phi(z_n^\varepsilon(0))\,dx + R_n.\end{aligned} \tag{5.6}$$

Our purpose is to recover an a priori estimate for z_n^ε. We begin applying Young's inequality with $\left(\frac{2}{q}, \frac{2}{2-q}\right)$ to the first integral in the right hand side of (5.6) obtaining

$$\begin{aligned}c_1 e^{(q-1)T} \iint_{Q_t} |\nabla G_k(z_n^\varepsilon)|^q \left[1 - \frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right] dx\,ds \\ \leq \frac{\mu\alpha}{2} \iint_{Q_t} \frac{|\nabla G_k(z_n^\varepsilon)|^2}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\,dx\,ds + c \iint_{Q_t} (1+G_k(z_n^\varepsilon))^{\frac{q(\mu+1)}{2-q}} \left[1 - \frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right]^{\frac{2}{2-q}} dx\,ds.\end{aligned}$$

We observe that, since $\mu < 1$, it holds that $1 - \frac{1}{(1+G_k(z_n^\varepsilon))^\mu} \leq \frac{G_k(z_n^\varepsilon)}{1+G_k(z_n^\varepsilon)}$, and since $\frac{2}{2-q} > 1$, it follows $\left(\frac{G_k(z_n^\varepsilon)}{1+G_k(z_n^\varepsilon)}\right)^{\frac{2}{2-q}} \leq G_k(z_n^\varepsilon)$ and thus we deduce the uniform boundedness

$$\begin{aligned}\int_\Omega \Phi(z_n^\varepsilon(t))\,dx + \mu\alpha \iint_{Q_t} \frac{|\nabla G_k(z_n^\varepsilon)|^2}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\,dx\,ds \\ \leq ce^{(q-1)T}\iint_{Q_t} |G_k(z_n^\varepsilon)|\,dx\,ds + \iint_{Q_t} |\tilde f|\chi_{\{|f|>k\}}\,dx\,ds + \int_\Omega \Phi(z_n^\varepsilon(0))\,dx + R_n,\end{aligned}$$

since, as already observed, the asymptotic condition (2.7) takes the place of (2.33) in the proof that $\lim_{n\to\infty} R_n = 0$. In particular, this means that the energy term above is uniformly bounded in n.

Furthermore, since $q\dfrac{\mu+1}{2-q} = q\dfrac{N+1}{N}$ is the Gagliardo-Nirenberg exponent associated to the spaces $L^2(0,T;H_0^1(\Omega))\cap L^\infty(0,T;L^{\frac{2}{1-\mu}}(\Omega))$, (see Theorem 3.1) we have that

$$\begin{aligned}\iint_{Q_T} (1+G_k(z_n^\varepsilon))^{\frac{q(\mu+1)}{2-q}}\left[1-\frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right]^{\frac{2}{2-q}}\,dx\,dt \leq \iint_{Q_T}(1+G_k(z_n^\varepsilon))^{\frac{q(\mu+1)}{2-q}-1}G_k(z_n^\varepsilon)\,dx\,dt \\ \leq c_{GN}\|G_k(z_n^\varepsilon)\|^{\frac{p}{N}}_{L^\infty(0,T;L^1(\Omega))}\iint_{Q_T}\frac{|\nabla G_k(z_n^\varepsilon)|^2}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\,dx\,dt\,.\end{aligned}$$

Then (5.6) becomes

$$\begin{aligned}\int_\Omega \Phi(z_n^\varepsilon(t))\,dx + \frac{\mu\alpha}{2}\iint_{Q_t}\frac{|\nabla G_k(z_n^\varepsilon)|^2}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\,dx\,ds \\ \leq \bar c\|G_k(z_n^\varepsilon)\|^{\frac{2-q}{N}}_{L^\infty(0,T;L^1(\Omega))}\iint_{Q_T}\frac{|\nabla G_k(z_n^\varepsilon)|^2}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\,dx\,dt + \iint_{Q_t}|\tilde f|\chi_{\{|f|>k\}}\,dx\,ds + \int_\Omega G_k(v_0)\,dx + \omega_n.\end{aligned}$$

We observe that the function $\Phi(z)$ can be estimated from below as $\Phi(w) > C_1\min\{G_k(w), G_k(w)^2\}$, for a certain $C_1 > 0$, from which

$$\begin{aligned}\int_\Omega G_k(z_n^\varepsilon(t))\,dx &\leq \int_{\{G_k(z_n^\varepsilon(t))>1\}} G_k(z_n^\varepsilon(t))\,dx + |\Omega|^{\frac12}\left(\int_{\{G_k(z_n^\varepsilon(t))\leq 1\}} G_k(z_n^\varepsilon(t))^2\,dx\right)^{\frac12} \\ &\leq \frac{1}{C_1}\int_\Omega \Phi(z_n^\varepsilon(t))\,dx + \left(\frac{|\Omega|}{C_1}\right)^{\frac12}\left(\int_\Omega \Phi(z_n^\varepsilon(t))\,dx\right)^{\frac12}.\end{aligned} \tag{5.7}$$

We fix a small value δ_0 so that the equality $\dfrac{\mu\alpha}{2} = \bar c(C_0\delta_0^{\frac12})^{\frac{p-q}{N}}$ holds for $C_0 = 2\max\left\{\dfrac{1}{C_1}, \left(\dfrac{|\Omega|}{C_1}\right)^{\frac12}\right\}$. Moreover, for $\delta < \delta_0$, we let k_0 large enough so that

$$\int_\Omega G_k(v_0)\,dx + \iint_{Q_t}|\tilde f|\chi_{\{|\tilde f|>k\}}\,dx\,ds < \delta \qquad \forall k \geq k_0 \tag{5.8}$$

and we define

$$T^* := \sup\{\tau > 0 : \|G_k(z_n^\varepsilon(s))\|_{L^1(\Omega)} \leq C_0\delta^{\frac12},\ \forall\, s \leq \tau\} \qquad \forall k \geq k_0.$$

Notice that $T^* > 0$ thanks to the regularity $u, v \in C([0,T];L^1(\Omega))$. We underline again that such a continuity regularity implies that T^* continuously depends on n.
The above choice of δ_0 and (5.8) imply

$$\int_\Omega \Phi(z^\varepsilon(t))\,dx \leq \int_\Omega G_k(v_0)\,dx + \iint_{Q_t}|\tilde f|\chi_{\{|\tilde f|>k\}}\,dx\,ds < \delta \qquad \forall k \geq k_0,\ \forall t \leq T^*. \tag{5.9}$$

Therefore, by definition of C_1 and C_0 and thanks to (5.7), we obtain

$$\int_\Omega G_k(z_n^\varepsilon(t))\,dx < C_0\delta^{\frac12} \quad \forall t \leq T^*. \tag{5.10}$$

By the continuity regularity $u, v \in C([0,T];L^1(\Omega))$ and (5.10) we deduce that $T^* = T$, since if $T^* < T$ then (5.10) would be in contrast with the definition of T^* and since $u, v \in C([0,T];L^1(\Omega))$.
Once we have (5.10) for $T = T^*$ then we have

$$\int_\Omega z_n^\varepsilon(t)\,dx \leq \int_\Omega G_k(z_n^\varepsilon(t))\,dx + \int_\Omega T_k(z_n^\varepsilon(t))\,dx \leq C_0\delta^{\frac12} + k$$

which, letting $n\to\infty$ and recalling the definition of z_n in (5.2), leads to

$$\int_\Omega \big(u(t) - (1-\varepsilon)v(t)\big)\,dx \leq \varepsilon e^t(C_0\delta^{\frac12} + k)$$

and the proof follows once we let $\varepsilon \to 0$. □

5.2. **The case** $2 < p < N$**.** Here we prove our results via the "convexity" method.

Proof of Theorem 2.13. We want to follow the first part of Theorem 2.8. In order to do it, we recall the definitions of u_n, v_n in (4.1) and consider the renormalized formulations in (2.6). We focus on the one related to the supersolution v: we consider $S(v) = v_n$ and multiply its inequality by $(1-\varepsilon)^{p-1}$, we get

$$\begin{aligned}\int_\Omega (1-\varepsilon)^{p-1} v_n(t)\varphi(t)\,dx + \iint_{Q_t} A(x)|\nabla(1-\varepsilon)v_n|^{p-2}\nabla((1-\varepsilon)v_n)\cdot\nabla\varphi\,dx\,ds \\ \geq (1-\varepsilon)^{p-1}\iint_{Q_t} H(s,x,\nabla v_n)\varphi\,dx\,ds \\ -\iint_{Q_t} A(x)\big[\theta_n(v)|\nabla((1-\varepsilon)v)|^{p-2}\nabla((1-\varepsilon)v) - |\nabla((1-\varepsilon)v_n)|^{p-2}\nabla((1-\varepsilon)v_n)\big]\cdot\nabla\varphi\,dx\,ds \\ -(1-\varepsilon)^{p-1}\iint_{Q_t} A(x)\nabla v\cdot\nabla v\,\theta_n'(v)\varphi\,dx\,ds + (1-\varepsilon)^{p-1}\iint_{Q_t}\big[H(s,x,\nabla v)\theta_n(v) - H(s,x,\nabla v_n)\big]\varphi\,dx\,ds \\ +(1-\varepsilon)^{p-1}\int_\Omega v_n(0)\varphi(0)\,dx,\end{aligned}$$

Then, rescaling in time this last inequality and defining v_n^ε as

$$v_n^\varepsilon(t,x) = (1-\varepsilon)v_n((1-\varepsilon)^{p-2}t, x),$$

we obtain

$$\begin{aligned}\int_\Omega v_n^\varepsilon(t)\varphi(t)\,dx + \iint_{Q_t^\varepsilon} A(x)|\nabla v_n^\varepsilon|^{p-2}\nabla v_n^\varepsilon\cdot\nabla\varphi\,dx\,ds \geq (1-\varepsilon)^{p-1}\iint_{Q_t^\varepsilon} H\left((1-\varepsilon)^{p-2}s, x, \frac{\nabla v_n^\varepsilon}{1-\varepsilon}\right)\varphi\,dx\,ds \\ -\iint_{Q_t^\varepsilon}\big[A(x)\left(\theta_n(v)|\nabla((1-\varepsilon)v)|^{p-2}\nabla((1-\varepsilon)v) - |\nabla v_n^\varepsilon|^{p-2}\nabla v_n^\varepsilon\right)\big]\cdot\nabla\varphi\,dx\,ds \\ -(1-\varepsilon)^{p-1}\iint_{Q_t^\varepsilon} A(x)\nabla v\cdot\nabla v\,\theta_n'(v)\varphi\,dx\,ds + (1-\varepsilon)^{p-1}\iint_{Q_t^\varepsilon}\left[H(s,x,\nabla v)\theta_n(v) - H(s,x,\nabla v_n)\right]\varphi\,dx\,ds \\ +\int_\Omega v_n^\varepsilon(0)\varphi(0)\,dx,\end{aligned} \tag{5.11}$$

where $Q_t^\varepsilon = \left(0, \frac{t}{(1-\varepsilon)^{p-2}}\right)\times\Omega$ for $0\le t\le T$. We want to take into account the difference between

$$\begin{aligned}\int_\Omega u_n(t)\varphi(t)\,dx + \iint_{Q_t} A(x)|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi\,dx\,ds \le \iint_{Q_t} H(s,x,\nabla u_n)\varphi\,dx\,ds \\ -\iint_{Q_t} A(x)\nabla u\cdot\nabla u\,\theta_n'(u)\varphi\,dx\,ds - \iint_{Q_t} A(x)\left(\theta_n(u)|\nabla u|^{p-2}\nabla u - |\nabla u_n|^{p-2}\nabla u_n\right) \\ +\iint_{Q_t}\left[H(s,x,\nabla u)\theta_n(u) - H(s,x,\nabla u_n)\right]\varphi\,dx\,ds + \int_\Omega u_n(0)\varphi(0)\,dx\end{aligned}$$

and (5.11): to this aim, we restrict the integrals to the time interval $0\le t\le T$ since (5.11) holds in $Q_t^\varepsilon\supset Q_t$.

As in the previous cases, our aim is to write an inequality solved by the following function

$$z_n^\varepsilon(t,x) = \frac{e^{-\lambda t}}{\varepsilon}\big(u_n(t,x) - v_n^\varepsilon(t,x)\big) - e^{-\lambda t}\bar v_0 - Mt \tag{5.12}$$

where $M>0$ has been defined in (2.39), $\lambda>0$ to be fixed and $\bar v_0$ (the upper bound of v_0) is assumed, without loss of generality, to be positive

$$\begin{aligned}\int_\Omega z_n^\varepsilon(t)\varphi(t)\,dx + \iint_{Q_t}(\lambda z_n^\varepsilon + M)\varphi\,dx\,ds + \iint_{Q_t}\frac{e^{-\lambda t}}{\varepsilon}A(x)\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla v_n^\varepsilon|^{p-2}\nabla v_n^\varepsilon\right)\cdot\nabla\varphi\,dx\,ds \\ \le \iint_{Q_t}\frac{e^{-\lambda t}}{\varepsilon}\left[H(s,x,\nabla u_n) - (1-\varepsilon)^{p-1}H\left((1-\varepsilon)^{p-2}s, x, \frac{\nabla v_n^\varepsilon}{1-\varepsilon}\right)\right]\varphi\,dx\,ds + \int_\Omega z_n^\varepsilon(0)\varphi(0)\,dx + R_n\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}R_n = &-\iint_{Q_t} \frac{e^{-\lambda t}}{\varepsilon} A(x)\big[\theta_n(u)|\nabla u|^{p-2}\nabla u - |\nabla u_n|^{p-2}\nabla u_n\big]\cdot\nabla\varphi\,dx\,ds\\
&-\iint_{Q_t} \theta_n'(u)A(x)|\nabla u|^{p-2}\nabla u\cdot\nabla u\,\varphi\,dx\,ds + \iint_{Q_t}\big[H(s,x,\nabla u)\theta_n(u) - H(s,x,\nabla u_n)\big]\varphi\,dx\,ds\\
&-\iint_{Q_t} \frac{e^{-\lambda t}}{\varepsilon} A(x)\big[\theta_n(v((1-\varepsilon)^{p-2}t,x))|\nabla v^\varepsilon|^{p-2}\nabla v^\varepsilon - |\nabla v_n^\varepsilon|^{p-2}\nabla v_n^\varepsilon\big]\cdot\nabla\varphi\,dx\,ds\\
&-(1-\varepsilon)^{-1}\iint_{Q_t}\theta_n'(v((1-\varepsilon)^{p-2}t,x))A(x)|\nabla v^\varepsilon|^{p-2}\nabla v^\varepsilon\cdot\nabla v\,\varphi\,dx\,ds\\
&+\iint_{Q_t}\big[H((1-\varepsilon)^{p-2}s,x,\nabla v^\varepsilon)\theta_n(v((1-\varepsilon)^{p-2}t,x)) - H((1-\varepsilon)^{p-2}s,x,\nabla v_n^\varepsilon)\big]\varphi\,dx\,ds.\end{aligned}$$

Let us define, for any $\mu>0$ and $a\geq 1$, the following function:

$$\Psi_{a,\mu}(v) = \int_0^{G_k(v)} (\mu+|w|)^{\frac{\sigma-2}{a}}\,dw. \tag{5.14}$$

Then, we set

$$\varphi(z_n^\varepsilon) = \Psi_{1,\mu}(z_n^\varepsilon) = \int_0^{G_k(z_n^\varepsilon)} (\mu+|w|)^{\sigma-2}\,dw$$

in (5.13), with $k\geq \bar v_0$ (in order to have $G_k(z_n^\varepsilon(0,x))\equiv 0$ in Ω) in (5.14) so that, thanks to (2.39), we get

$$\begin{aligned}&\int_\Omega \Phi(z_n^\varepsilon(t))\,dx + \lambda\iint_{Q_t} G_k(z_n^\varepsilon)\left(\int_0^{G_k(z_n^\varepsilon)}(\mu+|w|)^{\sigma-2}\,dw\right)dx\,ds + \frac{\alpha}{2}\varepsilon^{p-2}\iint_{Q_t}|\nabla\Psi_{p,\mu}(G_k(z_n^\varepsilon))|^p\,dx\,ds\\
&\qquad+\frac{\alpha}{2}\iint_{Q_t}|\nabla\Psi_{2,\mu}(G_k(z_n^\varepsilon))|^2\left[|\nabla u_n|^{p-2}+|\nabla v_n^\varepsilon|^{p-2}\right]dx\,ds\\
&\qquad\leq c_1 e^{\lambda(q-1)T}\iint_{Q_t}|\nabla G_k(z_n^\varepsilon)|^q\left(\int_0^{G_k(z_n^\varepsilon)}(\mu+|w|)^{\sigma-2}\,dw\right)dx\,ds\\
&\quad+c_2 e^{\frac{p-2}{2}\lambda T}\iint_{Q_t}|\nabla G_k(z_n^\varepsilon)|\left[|\nabla u_n|^{\frac{p-2}{2}}+|\nabla v_n^\varepsilon|^{\frac{p-2}{2}}\right]\left(\int_0^{G_k(\hat z_n^\varepsilon)}(\mu+|w|)^{\sigma-2}\,dw\right)dx\,ds + R_n\,.\end{aligned} \tag{5.15}$$

The proof $\lim_{n\to\infty} R_n = 0$ follows using the analogous one contained in Theorem 2.10 with $\ell=0$ (see (2.38)), changing (2.14) with (2.33) and taking advantage of the definition of $\varphi(\cdot)$.

We first deal with the latter integral in the right hand side. The definition of $|\nabla\Psi_{2,\mu}(G_k(z_n^\varepsilon))|$ and Young's inequality yield to

$$\begin{aligned}&c_2 e^{\frac{p-2}{2}\lambda T}\iint_{Q_t}|\nabla G_k(z_n^\varepsilon)|\left[|\nabla u_n|^{\frac{p-2}{2}}+|\nabla v_n^\varepsilon|^{\frac{p-2}{2}}\right]\left(\int_0^{G_k(z_n^\varepsilon)}(\mu+|w|)^{\sigma-2}\,dw\right)dx\,ds\\
&\leq c_2 e^{\frac{p-2}{2}\lambda T}\iint_{Q_t}|\nabla\Psi_{2,\mu}(G_k(z_n^\varepsilon))|\left[|\nabla u_n|^{\frac{p-2}{2}}+|\nabla v_n^\varepsilon|^{\frac{p-2}{2}}\right]\Psi_{2,\mu}(G_k(z_n^\varepsilon))\,dx\,ds\\
&\leq\frac{\alpha}{2}\iint_{Q_t}|\nabla\Psi_{2,\mu}(G_k(z_n^\varepsilon))|^2\left[|\nabla u_n|^{\frac{p-2}{2}}+|\nabla v_n^\varepsilon|^{\frac{p-2}{2}}\right]dx\,ds + c\iint_{Q_t}\left(\Psi_{2,\mu}(G_k(z_n^\varepsilon))\right)^2dx\,ds\\
&\leq\frac{\alpha}{2}\iint_{Q_t}|\nabla\Psi_{2,\mu}(G_k(z_n^\varepsilon))|^2\left[|\nabla u_n|^{\frac{p-2}{2}}+|\nabla\hat v_n^\varepsilon|^{\frac{p-2}{2}}\right]dx\,ds + \bar c\iint_{Q_t}G_k(z_n^\varepsilon)\left(\int_0^{G_k(z_n^\varepsilon)}(\mu+|w|)^{\sigma-2}\,dw\right)dx\,ds.\end{aligned}$$

Then, we use such a estimate to prove that, having also (2.33), provides us with the uniform bound in n of $(1+G_k(z_n^\varepsilon))^{\gamma-1}G_k(z_n^\varepsilon)$ in $L^p(0,T;W_0^{1,p}(\Omega))$. Indeed, (5.15) becomes

$$\begin{aligned}&\int_\Omega\Phi(z_n^\varepsilon(t))\,dx + \frac{\alpha}{2}\varepsilon^{p-2}\iint_{Q_t}|\nabla\Psi_{p,\mu}(G_k(z_n^\varepsilon))|^p\,dx\,ds\\
&\leq c_1 e^{(q-1)T}\iint_{Q_t}|\nabla G_k(z_n^\varepsilon)|^q\left(\int_0^{G_k(z_n^\varepsilon)}(\mu+|w|)^{\sigma-2}\,dw\right)dx\,ds + \omega_n\end{aligned}$$

provided $\lambda \geq \bar{c}$, and since

$$\begin{aligned}
&\iint_{Q_t} |\nabla G_k(z_n^\varepsilon)|^q \left(\int_0^{G_k(z_n^\varepsilon)} (\mu+|w|)^{\sigma-2}\,dw\right) dx\,ds \\
&\leq \iint_{Q_t} |\nabla \Psi_{p,\mu}(G_k(z_n^\varepsilon))|^q \left(\int_0^{G_k(z_n^\varepsilon)} (\mu+|w|)^{(\sigma-2)\frac{p-q}{p}}\,dw\right) dx\,ds \\
&\leq \iint_{Q_t} |\nabla \Psi_{p,\mu}(G_k(z_n^\varepsilon))|^q |\Psi_{p,\mu}(G_k(z_n^\varepsilon))|^{p-q} G_k(z_n^\varepsilon)^{q-(p-1)}\,dx\,ds
\end{aligned} \tag{5.16}$$

by definition of $\Psi_{p,\mu}(\cdot)$ and Hölder's inequality with indices $\left(\frac{1}{p-q}, \frac{1}{q-(p-1)}\right)$, we have that

$$\begin{aligned}
&\int_\Omega \Phi(z_n^\varepsilon(t))\,dx + \frac{\alpha}{2}\varepsilon^{p-2} \iint_{Q_t} |\nabla \Psi_{p,\mu}(G_k(z_n^\varepsilon))|^p\,dx\,ds \\
&\leq c_1 \iint_{Q_t} |\nabla \Psi_{p,\mu}(G_k(z_n^\varepsilon))|^q |\Psi_{p,\mu}(G_k(z_n^\varepsilon))|^{p-q} G_k(z_n^\varepsilon)^{q-(p-1)}\,dx\,ds + \omega_n.
\end{aligned} \tag{5.17}$$

An application of Young's inequality with $\left(\frac{p}{q}, \frac{p}{p-q}\right)$ implies

$$\int_\Omega \Phi(z_n^\varepsilon(t))\,dx + \frac{\alpha}{4}\varepsilon^{p-2} \iint_{Q_t} |\nabla \Psi_{p,\mu}(G_k(z_n^\varepsilon))|^p\,dx\,ds \leq c \iint_{Q_t} |\Psi_{p,\mu}(G_k(z_n^\varepsilon))|^p G_k(z_n^\varepsilon)^{\frac{p(q-(p-1))}{p-q}}\,dx\,ds + \omega_n.$$

The uniform boundedness of the right hand side above is due to the fact that $p\gamma + \frac{p(q-(p-1))}{p-q} = p\frac{N\beta+\sigma}{N}$ and that $(u-v) \in L^{p\frac{N+\sigma}{N}}(Q_T)$ by (2.33).
We continue applying once more the Hölder inequality with indices $\left(\frac{p}{q}, \frac{p^*}{p-q}, \frac{N}{p-q}\right)$ and also Sobolev's embedding, so we finally get

$$\begin{aligned}
&\iint_{Q_t} |\nabla \Psi_{p,\mu}(G_k(z_n^\varepsilon))|^q |\Psi_{p,\mu}(G_k(z_n^\varepsilon))|^{p-q} G_k(z_n^\varepsilon)^{q-(p-1)}\,dx\,ds \\
&\leq c \sup_{s\in(0,t)} \|G_k(z_n^\varepsilon(s))\|_{L^\sigma(\Omega)}^{q-p+1} \int_0^t \|\nabla \Psi_{p,\mu}(G_k(z_n^\varepsilon(s)))\|_{L^p(\Omega)}^p\,ds
\end{aligned}$$

where $c = c(\bar{\gamma}, N, q, T)$ and finally deduce that

$$\begin{aligned}
&\int_\Omega \Phi(z_n^\varepsilon(t))\,dx + \frac{\alpha}{2}\varepsilon^{p-2} \iint_{Q_t} |\nabla \Psi_{p,\mu}(G_k(z_n^\varepsilon))|^p\,dx\,ds \\
&\leq c \sup_{s\in(0,t)} \|G_k(z_n^\varepsilon(s))\|_{L^\sigma(\Omega)}^{q-p+1} \int_0^t \|\nabla \Psi_{p,\mu}(G_k(z_n^\varepsilon(s)))\|_{L^p(\Omega)}^p\,ds + \omega_n.
\end{aligned}$$

Then, the above uniform boundedness in n on the difference between sub/supersolutions allow us to let $n \to \infty$ getting

$$\int_\Omega \Phi(z^\varepsilon(t))\,dx + \frac{\alpha}{2}\varepsilon^{p-2} \iint_{Q_t} |\nabla \Psi_{p,\mu}(G_k(z^\varepsilon))|^p\,dx\,ds \leq c \sup_{s\in(0,t)} \|G_k(z^\varepsilon(s))\|_{L^\sigma(\Omega)}^{q-p+1} \iint_{Q_t} |\nabla \Psi_{p,\mu}(G_k(z^\varepsilon))|^p\,dx\,ds.$$

We now reason as in Theorem 2.8 and, being $\Phi(w) \longrightarrow \frac{|w|^\sigma}{\sigma(\sigma-1)}$ as $\mu \to 0$ and thanks to Lemma 3.3, we have that $G_k(z^\varepsilon) \equiv 0$ in Q_t. In particular, this means that

$$e^{-\lambda t}\Big(u(t,x) - (1-\varepsilon)v\big((1-\varepsilon)^{p-2}t, x\big) - \varepsilon\bar{v}_0\Big) - \varepsilon M t \leq \varepsilon k$$

and letting ε vanishes we deduce that $u(t,x) \leq v(t,x)$ in Q_T, as desired.

□

Proof of Theorem 2.14. We set $\lambda = 1$ in (5.12) and take

$$\varphi(z_n^\varepsilon) = 1 - \frac{1}{(1+G_k(z_n^\varepsilon))^\mu}, \quad \text{with} \quad \mu = (p-q)\frac{N+1}{N} - 1 > p-1$$

in (5.13), so that

$$
\begin{aligned}
&\int_\Omega \Phi(z_n^\varepsilon(t))\,dx + \frac{\mu\alpha}{2}\varepsilon^{p-2}\iint_{Q_t}\frac{|\nabla G_k(z_n^\varepsilon)|^p}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\,dx\,ds\\
&+\frac{\mu\alpha}{2}\iint_{Q_t}\frac{|\nabla G_k(z_n^\varepsilon)|^2}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\left[|\nabla u_n|^{p-2}+|\nabla v_n^\varepsilon|^{p-2}\right]\,dx\,ds\\
&\le c_1 e^{(q-1)\lambda T}\iint_{Q_t}|\nabla G_k(z_n^\varepsilon)|^q\left[1-\frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right]\,dx\,ds\\
&+c_2 e^{\frac{p-2}{2}\lambda T}\iint_{Q_t}|\nabla (G_k(z_n^\varepsilon))|\left[|\nabla u_n|^{\frac{p-2}{2}}+|\nabla v_n^\varepsilon|^{\frac{p-2}{2}}\right]\left[1-\frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right]\,dx\,ds+R_n
\end{aligned}
\tag{5.18}
$$

thanks to (2.39).

The proof that $\lim_{n\to\infty} R_n = 0$ follows reasoning as in Theorem 2.11 (see also Theorem 2.9), so we skip it.

We start estimating the first integral in the right hand side above through Young's inequality with indices $\left(\frac{p}{q},\frac{p}{p-q}\right)$, getting

$$
\begin{aligned}
&c_1 e^{(q-1)\lambda T}\iint_{Q_t}|\nabla G_k(z_n^\varepsilon)|^q\left[1-\frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right]\,dx\,ds\\
&\le \frac{\mu\alpha}{4}\varepsilon^{p-2}\iint_{Q_t}\frac{|\nabla (G_k(z_n^\varepsilon))|^p}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\,dx\,ds + c\iint_{Q_t}(1+G_k(z_n^\varepsilon))^{\frac{p(\mu+1)}{p-q}}\,dx\,ds.
\end{aligned}
$$

As far as the second integral in the right hand side of (5.18), applying again Young's inequality with indices $(2,2)$ we get

$$
\begin{aligned}
&c_2 e^{\frac{p-2}{2}\lambda T}\iint_{Q_t}|\nabla G_k(z_n^\varepsilon)|\left[|\nabla u_n|^{\frac{p-2}{2}}+|\nabla v_n^\varepsilon|^{\frac{p-2}{2}}\right]\left[1-\frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right]\,dx\,ds\\
&\le \frac{\mu}{2}\iint_{Q_t}\frac{|\nabla G_k(z_n^\varepsilon)|^2}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\left[|\nabla u_n|^{p-2}+|\nabla v_n|^{p-2}\right]\,dx\,ds + c\iint_{Q_t}(1+G_k(z_n^\varepsilon))^{\mu+1}\,dx\,ds.
\end{aligned}
\tag{5.19}
$$

Using the previous estimates into (5.18) we obtain

$$
\begin{aligned}
&\int_\Omega \Phi(z_n^\varepsilon(t))\,dx + \frac{\mu\alpha}{4}\varepsilon^{p-2}\iint_{Q_t}\frac{|\nabla (G_k(z_n^\varepsilon))|^p}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\,dx\,ds\\
&\le c\left[\iint_{Q_t}(1+G_k(z_n^\varepsilon))^{\frac{p(\mu+1)}{p-q}}\,dx\,ds + \iint_{Q_t}(1+G_k(z_n^\varepsilon))^{\mu+1}\,dx\,ds\right]+\omega_n.
\end{aligned}
$$

Then, since having $p>2$ and $q>\frac{p(N+1)-N}{N+2}$ imply that both $\frac{p(\mu+1)}{p-q}$ are $\mu+1$ smaller than $q\frac{N+1}{N}$ (see Lemma 3.5 and Theorem 3.1), the right hand side in the above inequality is uniformly bounded with respect to n.

Now, let us focus on the right hand side of (5.18). We apply Hölder's inequality with $\left(\frac{p}{q},\frac{p}{p-q}\right)$ on the integral involving the q power of the gradient, getting

$$
\begin{aligned}
&c_1 e^{(q-1)\lambda T}\iint_{Q_t}|\nabla G_k(z_n^\varepsilon)|^q\left[1-\frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right]\,dx\,ds\\
&\le c\left(\iint_{Q_t}(1+G_k(z_n^\varepsilon))^{\frac{q(\mu+1)}{p-q}}\left[1-\frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right]^{\frac{p}{p-q}}\,dx\,ds\right)^{\frac{p-q}{p}}\left(\iint_{Q_t}\frac{|\nabla G_k(z_n^\varepsilon)|^p}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\,dx\,ds\right)^{\frac{q}{p}}.
\end{aligned}
$$

We observe that the value $\nu = p\frac{N+\frac{p}{p-\mu-1}}{N}$ corresponds to the Gagliardo-Nirenberg regularity exponent for the spaces

$$
L^\infty(0,T;L^{\frac{p}{p-\mu-1}}(\Omega))\cap L^p(0,T;W_0^{1,p}(\Omega)).
$$

We justify such a gradient regularity recalling Lemma 3.5. Then, we have

$$
\begin{aligned}
\iint_{Q_T}(1+G_k(z_n^\varepsilon))^{\frac{q(\mu+1)}{p-q}}\left[1-\frac{1}{(1+G_k(z_n^\varepsilon))^\mu}\right]^{\frac{p}{p-q}}\,dx\,dt &\le \int_0^T\left\|(1+G_k(z_n^\varepsilon))^{-\frac{\mu+1}{p}}G_k(z_n^\varepsilon)\right\|_{L^\nu(\Omega)}^\nu\,dt\\
&\le c_{GN}\|G_k(z_n^\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))}^{\frac{p}{N}}\iint_{Q_T}\frac{|\nabla G_k(z_n^\varepsilon)|^p}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\,dx\,dt
\end{aligned}
$$

thanks to (3.2). Furthermore, since the last integral in (5.19) can be estimated by

$$\iint_{Q_t}(1+G_k(z_n^\varepsilon))^\mu G_k(z_n^\varepsilon)\,dx\,ds \le \iint_{Q_t}(1+G_k(z_n^\varepsilon))^{q\frac{N+1}{N}-1}G_k(z_n^\varepsilon)\,dx\,ds$$
$$\le c\|G_k(z_n^\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))}^{\frac{p-q}{N}}\iint_{Q_T}\frac{|\nabla G_k(z_n^\varepsilon)|^p}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\,dx\,dt$$

we are left with the study of

$$\int_\Omega \Phi(z_n^\varepsilon(t))\,dx+\frac{\mu\alpha}{2}\varepsilon^{p-2}\iint_{Q_t}\frac{|\nabla(G_k(z_n^\varepsilon))|^p}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\,dx\,ds \le c\|G_k(z_n^\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))}^{\frac{p-q}{N}}\iint_{Q_T}\frac{|\nabla G_k(z_n^\varepsilon)|^p}{(1+G_k(z_n^\varepsilon))^{\mu+1}}\,dx\,dt+\omega_n.$$

Then, letting $n\to\infty$ in the inequality above yields to

$$\int_\Omega \Phi(z^\varepsilon(t))\,dx+\frac{\mu\alpha}{2}\varepsilon^{p-2}\iint_{Q_t}\frac{|\nabla(G_k(z^\varepsilon))|^p}{(1+G_k(z^\varepsilon))^{\mu+1}}\,dx\,ds \le c\|G_k(z^\varepsilon)\|_{L^\infty(0,t;L^1(\Omega))}^{\frac{p-q}{N}}\iint_{Q_t}\frac{|\nabla G_k(z^\varepsilon)|^p}{(1+G_k(z^\varepsilon))^{\mu+1}}\,dx\,ds$$

We conclude reasoning as in the proof of Theorem 2.9, recalling Lemma 3.3 and letting $\varepsilon\to 0$.

□

Appendix A.

Our current goal is proving that one needs to consider sub/supersolutions belonging to the regularity class (2.13)–(2.14) in order to have a uniqueness result for problems of (1.2) type.

Here we use a result contained in [BASW1, Section 3] (see also [BASW2]), where it is proved that the Cauchy problem

$$\begin{cases} u_t-\Delta u=c_1|\nabla u|^q & \text{in } (0,T)\times\mathbb{R}^N,\\ u(0,x)=0 & \text{in } \mathbb{R}^N,\end{cases}\tag{A.1}$$

with $c_1>0$ and $q>1$ and $N\ge 2$ admits at least two solutions.

More precisely they prove that there exists a value $\alpha_0>0$ and a (unique) solution $U\in C^2(0,\infty)\cap C^1[0,\infty)$ of the following Cauchy problem:

$$\begin{cases} U''+\left(\frac{N-1}{y}+\frac{y}{2}\right)U'+kU+c_1|U'|^q=0 & \text{for } 0<y<\infty,\\ U'(0)=0,\\ U(0)=\alpha_0\end{cases}\tag{A.2}$$

that satisfies

$$U(y)=ce^{-\frac{y^2}{4}}y^{-\frac{N}{\sigma'}}\left(1+o(y^{-2})\right)\quad\text{as}\quad y\to\infty,\quad\text{with } \sigma=\frac{N(q-1)}{2-q},\tag{A.3}$$

$$U'(y)=-\frac{y}{2}U(y)\,(1+o(1))\quad\text{as}\quad y\to\infty\tag{A.4}$$

and

$$U\in L^j([0,\infty);y^{N-1}dy)\quad\text{for any}\quad 1\le j<\infty.\tag{A.5}$$

Consequently it is easy to see that both $u_1\equiv 0$ and $u_2(t,x)=t^{-\frac{N}{2\sigma}}U(|x|/\sqrt{t})$ solve (A.1).

We use such a result in order to show that the class of uniqueness (2.13)–(2.14) in the right one in order to have comparison (and thus uniqueness). Indeed we construct, for the following problem

$$\begin{cases} u_t-\Delta u=|\nabla u|^q+f(t,x) & \text{in } (0,T)\times B_R(0),\\ u(t,x)=0 & \text{on } (0,T)\times\partial B_R(0),\\ u(0,x)=0 & \text{in } B_R(0),\end{cases}$$

for a suitable choice of f smooth, a pair of solutions, whose only one belong to the class (2.13)–(2.14), while the other one is not regular enough.

In fact we have the following result.

Theorem A.1. *Let $2-\frac{N}{N+1}<q<2$, $R>0$ and let U be the positive solution of (A.2). Then the Cauchy-Dirichlet problem*

$$\begin{cases} v_t-\Delta v=c_1|\nabla v|^q+\left(t^{-\frac{N}{2\sigma}}U(R/\sqrt{t})\right)' & \text{in } (0,T)\times B_R(0),\\ v=0 & \text{on } (0,T)\times\partial B_R(0),\\ v(0,x)=0 & \text{in } B_R(0),\end{cases}\tag{A.6}$$

admits at least two solutions $v_{1,2}$ such that:

- *$v_1\in C([0,T];L^\infty(B_R(0)))$ and $|v_1|^{\frac{\sigma}{2}}\in L^2(0,T;H_0^1(B_R(0)))$, with $\sigma=\frac{N(q-1)}{2-q}$;*

- $v_2(t,x) = t^{-\frac{N}{2\sigma}}\big(U(|x|/\sqrt{t}) - U(R/\sqrt{t})\big)$, *and it satisfies*
$$v_2 \in C([0,T];L^\mu(B_R(0))) \quad \text{for any} \quad 1 \le \mu < \sigma \quad \text{but} \quad v_2 \notin C([0,T];L^\sigma(B_R(0))) \tag{A.7}$$
and
$$|v_2|^\beta \in L^2(0,T;H_0^1(B_R(0))), \quad \text{for any} \quad \beta < \frac{\sigma}{2}, \quad \text{but} \quad |v_2|^{\frac{\sigma}{2}} \notin L^2(0,T;H_0^1(B_R(0))). \tag{A.8}$$

Proof. We proceed observing that, thanks to (A.3)–(A.4), then $\left(t^{-\frac{N}{2\sigma}}U\left(\frac{R}{\sqrt{t}}\right)\right)' \in C^1([0,T])$ and thus (A.6) admits a solution v_1 such that $v_1 \in C([0,T];L^\sigma(\Omega))$ and $|v_1|^{\frac{\sigma}{2}} \in L^2(0,T;H_0^1(B_R(0)))$ (see [Ma]).

Then, we are left with the proofs of (A.7)–(A.8).

In order to prove that $\|v_2(t)\|_{L^\mu(B_R(0))} \to 0$, as $t \to 0^+$, for $\mu < \sigma$, we compute
$$\int_{B_R(0)} |v_2(t,x)|^\mu\,dx = t^{-\mu\frac{N}{2\sigma}} \int_0^R |U(r/\sqrt{t}) - U(R/\sqrt{t})|^\mu r^{N-1}\,dr = t^{\frac{N}{2}(1-\frac{\mu}{\sigma})} \int_0^{\frac{R}{\sqrt{t}}} |U(y) - U(R/\sqrt{t})|^\mu y^{N-1}\,dy$$
$$\le t^{\frac{N}{2}(1-\frac{\mu}{\sigma})} \int_0^\infty |U(y) - U(R/\sqrt{t})|^\mu y^{N-1}\,dy \le ct^{\frac{N}{2}(1-\frac{\mu}{\sigma})}$$
where the last inequality follows from (A.5). Then, if $\mu < \sigma$, we have $\frac{N}{2}\left(1-\frac{\mu}{\sigma}\right) > 0$ which implies that the right hand side of the above inequality vanishes as $t \to 0^+$. If, on the contrary, we set $\mu = \sigma$ then the integral above becomes
$$\int_{B_R(0)} |v_2(t,x)|^\sigma\,dx = \int_0^{\frac{R}{\sqrt{t}}} |U(y) - U(R/\sqrt{t})|^\sigma y^{N-1}\,dy$$
which is bounded from below, thanks to (A.5), by a positive constant.

In order to prove (A.8), we observe that
$$\begin{aligned}\int_0^T \int_{B_R(0)} |\nabla |v_2|^\beta|^2\,dx\,dt &= \int_0^T \int_0^R |\nabla |t^{-\frac{N}{2\sigma}} U(r/\sqrt{t})|^\beta|^2 r^{N-1}\,dr\,dt \\ &= \beta^2 \int_0^T \int_0^R t^{-\frac{N}{\sigma}\beta-1} U(r/\sqrt{t})^{2(\beta-1)} U'(r/\sqrt{t})^2 r^{N-1}\,dr\,dt \\ &= \beta^2 \int_0^T t^{-\frac{N}{\sigma}\beta-1+\frac{N}{2}} \int_0^{\frac{R}{\sqrt{t}}} U(y)^{2(\beta-1)} U'(y)^2 y^{N-1}\,dy\,dt.\end{aligned} \tag{A.9}$$
Then, recalling (A.3)–(A.4), we get
$$\int_0^T \int_{B_R(0)} |\nabla |v_2|^\beta|^2\,dx\,dt \le c\int_0^T t^{-\frac{N}{\sigma}\beta-1+\frac{N}{2}} \int_0^\infty U(y)^{2(\beta-1)} U'(y)^2 y^{N-1}\,dy\,dt$$
$$\le c\int_0^T t^{-\frac{N\beta}{\sigma}-1+\frac{N}{2}} \int_0^\infty e^{-\beta\frac{y^2}{2}} y^{-2N\beta(\frac{\sigma-1}{\sigma})+N+1}\,dy\,dt \le c\int_0^T t^{-\frac{N\beta}{\sigma}-1+\frac{N}{2}}\,dt\,,$$
which is finite if $\beta < \frac{\sigma}{2}$. On the other hand, if we suppose that $\beta = \frac{\sigma}{2}$, then (A.9) becomes
$$\int_0^T \int_{B_R(0)} |\nabla |v_2|^{\frac{\sigma}{2}}|^2\,dx\,dt = \frac{\sigma^2}{4} \int_0^T t^{-1} \int_0^{\frac{R}{\sqrt{t}}} U(y)^{\sigma-2} U'(y)^2 y^{N-1}\,dy\,dt$$
$$\ge \frac{\sigma^2}{4} \int_0^1 t^{-1} \int_0^R U(y)^{\sigma-2} U'(y)^2 y^{N-1}\,dy\,dt \ge c\int_0^1 t^{-1}\,dt = +\infty\,,$$
and the assertion follows. □

References

[1] B. Abdellaoui, A. Dall'Aglio, I. Peral; *Some remarks on elliptic problems with critical growth in the gradient* J. Differential Equations, **222** (2006), 21–62.

[2] B. Abdellaoui, A. Dall'Aglio, I. Peral; *Regularity and nonuniqueness results for parabolic problems arising in some physical models, having natural growth in the gradient* J. Math. Pures Appl. **90** (2008) 242–269.

[3] A.Alvino, M.F. Betta, A. Mercaldo; *Comparison principle for some classes of nonlinear elliptic equations*, J. Diff. Eq., **249** (2010) 3279–3290.

[4] G.Barles, F.Murat; *Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions* Arch. Rational Mech. Anal., **133** (1995), 77–101.

[5] G. Barles, F. Da Lio, *On the generalized Dirichlet problem for viscous Hamilton-Jacobi equations* J. Math. Pures Appl. **83** (2004), 53–75.

[6] G. Barles , A. Porretta, *Uniqueness for unbounded solutions to stationary viscous Hamilton-Jacobi equations*, Ann. Scuola Norm. Sup. di Pisa Cl. Sci., (5), **5** (2006), 107–136.

[7] M. Ben-Artzi, P. Souplet , F. Weissler, *The local theory for viscous Hamilton-Jacobi equations in Lebesgue spaces*, J. Math. Pures Appli., **81** (2002), 343–378.

[8] M. Ben-Artzi, P. Souplet, F. Weissler, Sur la non-existence et la non-unicité des solutions du problème de Cauchy pour une équation parabolique semi-linéaire, Comptes Rendus de l'Académie des Sciences-Series I-Mathematics, 329.5 (1999), 371–376.

[9] P. Benilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre , J.L. Vázquez, *An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa, **22** (1995), 241–273.

[10] M.F. Betta, R. Di Nardo, A. Mercaldo, A. Perrotta, *Gradient estimates and comparison principle for some nonlinear elliptic equations*, Commun. Pure Appl. Anal. **14** (2015) 897–922.

[11] F. Betta, A. Mercaldo, F. Murat, M. Porzio, *Uniqueness of renormalized solutions to nonlinear elliptic equations with a lower order term and right-hand side in $L^1(\Omega)$. A tribute to J. L. Lions*, ESAIM Control Optim. Calc. Var., **8** (2002), 239–272.

[12] F. Betta, A. Mercaldo, F. Murat, M. Porzio, *Uniqueness results for nonlinear elliptic equations with a lower order term*, Nonlinear Anal. **63** (2005), 153–170.

[13] D. Blanchard , F. Murat, *Renormalised solutions of nonlinear parabolic problems with L^1 data: existence and uniqueness*, Proceedings of the Royal Society of Edinburgh, **127** (1997), 1137-1152.

[14] D. Blanchard , A. Porretta, *Stefan problems with nonlinear diffusion and convection*, J. Diff. Eq., **210** (2005), 383–428.

[15] L. Boccardo, A. Dall'Aglio, T. Gallouët , L. Orsina, *Nonlinear Parabolic Equations with Measure Data*, J. Func. An., **147** (1997), 237–258.

[16] M. Crandall, H. Ishii, P.L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bulletin of the American Mathematical Society (1992), Volume 27, Number 1.

[17] R. Di Nardo, F. Feo, O. Guibé, *Uniqueness of renormalized solutions to nonlinear parabolic problems with lower-order terms*, Proc. Roy. Soc. Edinburgh Sect. A **143** (2013), 1185–1208.

[18] F. Feo, *A remark on uniqueness of weak solutions for some classes of parabolic problems*, Ric. Mat., **63** (2014), S143–S155

[19] N. Grenon, F. Murat , A. Porretta, *Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms*, C. R. Acad. Sci. Paris, Ser. I, **342** (2006), 23–28.

[20] N. Grenon, F. Murat , A. Porretta, *A priori estimates and existence for elliptic equations with gradient dependent terms*, Ann. Sc. Norm. Super. Pisa Cl. Sci., **13** (2014), 137–205.

[21] T. Leonori, A. Porretta, *On the comparison principle for unbounded solutions of elliptic equations with first order terms*, Journal of Mathematical Analysis and Applications, in press.

[22] T. Leonori, A. Porretta, G. Riey, *Comparison principles for p-Laplace equations with lower order terms*, To appear in Ann. Mat. Pura Appl.

[23] M. Magliocca, *Existence results for a Cauchy-Dirichlet parabolic problem with a repulsive gradient term*, Preprint.

[24] A. Mercaldo, *A priori estimates and comparison principle for some nonlinear elliptic equations*, In: Magnanini R., Sakaguchi S., Alvino A. (eds) Geometric Properties for Parabolic and Elliptic PDE's. Springer INdAM Series, vol 2. (2013) Springer, Milano

[25] F. Petitta, *Renormalized solutions of nonlinear parabolic equations with general measure data*, Ann. Mat. Pura Appl., **187** (2008), 563–604.

[26] F. Petitta, A. Ponce , A. Porretta, *Diffuse measures and nonlinear parabolic equations*, J. Evol. Eq., **11** (2011), 861–905.

[27] A. Porretta, *Existence results for nonlinear parabolic equations via strong convergence of truncations*, Ann. Mat. Pura e Appl., **177** (1999) 143–172.

[28] A. Porretta, *On the comparison principle for p-Laplace type operators with first order terms*, in "On the notions of solution to nonlinear elliptic problems: results and developments", 459–497, Quad. Mat. 23, Dept. Math., Seconda Univ. Napoli, Caserta (2008)

[29] G. Stampacchia, Le probléme de Dirichlet pour les équations elliptiques du second ordre á coefficients discontinus, Ann. Inst. Fourier (Grenoble), **15** (1965), 189–258.

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