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Periodic projections of alternating knots

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A B S T R A C T

This paper is devoted to the proof of existence of q-periodic alternating projections of prime alternating q-periodic knots. The main tool is the Menasco-Thistlethwaite's Flyping Theorem.

Let K be an oriented prime alternating knot that is q-periodic with $q \geq 3$, i.e. that admits a rotation of order q as a symmetry. Then K has an alternating projection $\Pi(K)$ such that the rotational symmetry of K is visualized as a rotation of the projection sphere leaving $\Pi(K)$ invariant.

As an application, we obtain that the crossing number of a q-periodic alternating knot with $q \geq 3$ is a multiple of q. Furthermore we give an elementary proof that the knot $12a_{634}$ is not 3-periodic; our proof does not depend on computer calculations as in [11].

1. Introduction

For the general concepts of knot theory, we refer to the classical books on the subject, for example [19] and [14]. For pedagogical reasons, in this introduction we will define informally some of these concepts.

A *knot* is a circle embedded in S^3, a *link* is a disjoint union of knots. We restrict our attention to knots although some results are also valid for links. A knot is said to be *oriented* if we choose a direction to travel around it.

We consider two knots K_1 and K_1 *equivalent* if there exists an orientation-preserving homeomorphism of pairs $h : (S^3, K_1) \to (S^3, K_2)$. A knot is *tame* if in its equivalence class, there is a (finite) polygonal curve. We will only consider tame knots.

Since the origins of knot theory, one of the methods for studying links and knots is to use the projections on a sphere of S^3 (this is the case since the first publications on knot theory by P. G. Tait from 1877). A *projection* of a knot is its image by a central projection onto a sphere (called *projection sphere*), such that no three points of the knot project onto the same point. In a projection, at each double point, we will indicate, by the traditional graphical method, which segment "overpasses" and which one "underpasses".

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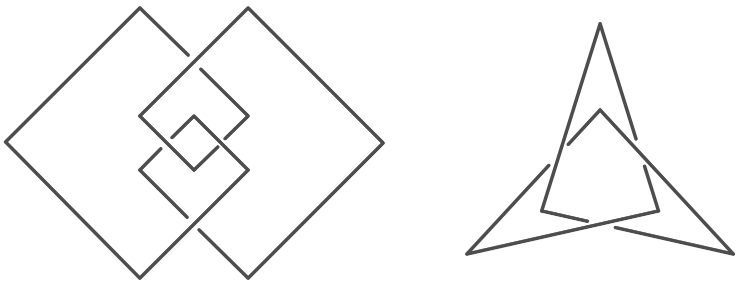

Fig. 1. 2-periodic and 3-periodic projections of the trefoil knot. Note that the 2-periodic projection is not alternating.

Alternating projections are a special type of projections that have captured the attention of knot theorists from the very beginning; a knot projection is *alternating* if as one travels along it, one alternatively overpasses, underpasses, overpasses, underpasses and so on at the crossing points. A knot is alternating if it has an alternating projection.

For any geometrical (or even mathematical) object, it is natural to study its symmetries, this has also been one of the classical problems in knot theory (see Chapter 8 on symmetries of knots in [14]). The simplest symmetries are certainly those given by a rotation around an axis of S^3. We will say that a knot K is *q-periodic* if there is a rotation of order q of S^3 that leaves K invariant (in the case $q = 2$, the axis of the rotation must not intersect the knot).

The easiest way to find symmetries of a knot is by studying that on its projections. A projection is *q-periodic* if there is a rotation of order q of the projection sphere leaving the projection of the knot invariant with its overpasses and underpasses. Determining whether a projection of a knot K is q-periodic is something that can be efficiently done, but the main problem is to determine for which of the infinite many projections of K the symmetries are to be studied. The following theorem implies that, for alternating knots, the possible q-periodicities with $q > 2$ can be "visualized" on alternating projections.

Visibility Theorem 3.1. *Let K be a prime alternating knot that is q-periodic with $q \geq 3$. Then K has a q-periodic alternating projection.*

Note that the condition $q \geq 3$ is clearly necessary. The simplest example of alternating 2-periodic knot without 2-periodic alternating projections is the trefoil knot (see Fig. 1), another more involved example is the knot 7_6 (see figures in [14]).

The Menasco-Thistlethwaite's Flyping Theorem (see Section 2.3.1 and [15]) states that alternating projections of the same link are essentially related by flypes (see Fig. 6). The Flyping Theorem and the Visibility Theorem reduce the task of finding q-periodicities ($q \geq 3$) of a knot to a process with a finite number of steps. The Flyping Theorem is also a fundamental ingredient in the proof of the Visibility Theorem.

There are other types of symmetries of knots that are well studied. One of the most popular symmetries is the achirality. A knot is *achiral* if there is a symmetry of the knot which is an orientation-reversing homeomorphism of S^3. The achirality of alternating knots is studied in [7] and [8].

We also obtain the following direct applications of the Visibility Theorem:

1. If Seifert's algorithm is applied on a q-periodic projection of an oriented knot, the resulting surface exhibits a q-periodic symmetry. Such a surface is called q-equivariant. The topological types of periodic homeomorphisms of bordered surfaces that are equivariant Seifert surfaces of periodic knots are studied in [4]. A. Edmonds [6] shows that if a knot K is of period q, then there is a q-equivariant Seifert surface for K, which has the genus of K. For a q-periodic oriented alternating knot K with $q \geq 3$, the strategy explained in the proof of Theorem 3.1 enables one to exhibit the realization of a q-equivariant surface from Seifert's algorithm which has the genus of K.

2. We obtain the following:

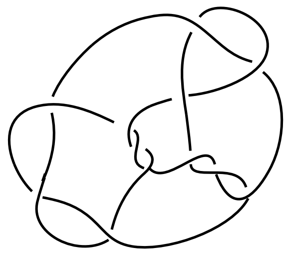

Fig. 2. The knot $12a_{634}$.

Proposition 3.5 *(Conjecture in Section 1.4 of [5]).* The crossing number of a prime q-periodic alternating knot with $q \geq 3$ is a multiple of q.

This result gives a direct method to remove prime knots with an incompatible crossing number to the list of possible q-periodic, $q \geq 3$. For instance, if a prime alternating knot K has crossing number m Proposition 3.5 implies directly that K has no periods $q > m$.

3. The proof of the Visibility Theorem 3.1 provides us a method to decide if a prime knot is q-periodic. As an example, we show that the knot $12a_{634}$ (see Fig. 2) is not 3-periodic.

We thank C. Livingstone to point out the existence of a computer proof of this fact by S. Jabuka and S. Naik [11].

4. Finally, we complete the analysis of the relation between adjacency graphs of Murasugi decompositions of alternating knots and q-periodicity started in [5] with:

Theorem 3.3. *Let K be a prime oriented alternating q-periodic knot with $q \geq 3$ and $A(K)$ its collection of Murasugi atoms. Then the adjacency graph of K admits an automorphism of order q and each atom of K is either q-periodic or it occurs a multiple of q times in $A(K)$.*

Acknowledgments

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1.1. Organization of the paper

For the study of the visibility of the periodicity of the alternating knots on alternating projections, we will call upon the canonical decomposition of knots projections recalled in Section 2, as it was done for the visibility of achirality of the alternating knots in [8]. The decomposition of a knot projection Π is carried out by a family of canonical Conway circles which decomposes (S^2, Π) into diagrams called *jewels* and *twisted band diagrams*; the arborescent part of Π is the union of the twisted band diagrams of Π. The decomposition of (S^2, Π) by the canonical Conway circles is a 2-dimensional version of the decomposition of Bonahon-Siebenmann [1] of (S^3, K). The notion of flype in alternating projections (Fig. 6) is at the heart of our analysis and lies completely in the arborescent part.

According to the Menasco-Thistlethwaite's Flyping theorem [15], two reduced alternating projections Π_1 and Π_2 of an equivalence class of an alternating knot K are related by a finite sequence of flypes, up to homeomorphisms of S^2 onto itself. Starting from the canonical decomposition of an alternating projection of Π of K, we associate canonical and essential structure trees (as recalled in §2) that do not depend on the choice of the alternating projection. The canonical and essential structure trees are invariants of the

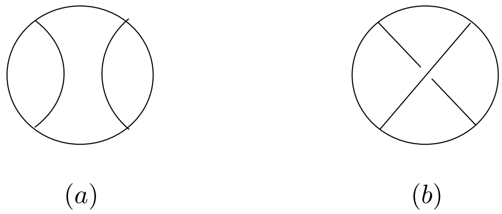

Fig. 3. (a) Trivial diagram. (b) A singleton.

equivalence class of an alternating knot. For example, for rational knots, their canonical structure tree is a linear tree with integer-weighted vertices and their essential structure tree is reduced to a vertex of rational weight.

In Section 3 we study how the q-periodicity acts on the essential Conway circles and on the structure trees. With the help of the Kerekjarto's theorem [3] and the Flyping Theorem of Menasco-Thistlethwaite we prove Theorem 3.1 and we obtain a q-periodic alternating projection by adjustments with flypes on any alternating projection of the knot.

Finally, we present some applications as mentioned in the introduction.

2. Canonical decomposition of a projection

In this section we do not assume that knots projections are alternating. A **projection** on S^2 is the image of a knot in S^3 by a generic projection onto S^2, hence a labeled graph with 4-valent crossing-vertices labeled to reflect under and over crossings.
In this paper the term "**diagram**" will be used to refer to a different object (see §2.1 below).

2.1. Diagrams

Let Σ be a compact connected planar surface embedded on the projection sphere S^2. We denote by $k+1$ the number of connected components of its boundary $\partial\Sigma$.

Definition 2.1. The pair $D = (\Sigma, \Gamma = \Pi \cap \Sigma)$ where Π is a knot projection is called a **diagram** if, for each connected component C of $\partial\Sigma$, $C \cap \Pi$ is composed exactly of 4 points that are not crossing points.

Remark 2.1. A (knot) projection Π on S^2 is a diagram (Σ, Π) where $\Sigma = S^2$.

Definition 2.2. (1) A **trivial diagram** is a diagram without crossings (see Fig. 3(a)).
(2) A **singleton** is a diagram with only a crossing (see Fig. 3(b)).

Definition 2.3. A **Haseman circle** of a diagram $D = (\Sigma, \Gamma)$ is a circle γ contained in the interior of Σ that intersects the projection Π exactly in 4 points away from crossing points. A Haseman circle is said to be **compressible** if γ bounds a disc Δ in Σ such that $(\Delta, \Gamma \cap \Delta)$ is either a trivial diagram or a singleton.

In what follows, Haseman circles are not compressible. We therefore only consider diagrams that are neither trivial diagrams nor singletons.

Definition 2.4. A **twisted band diagram** (TBD) is a diagram that is homeomorphic to the diagram in Fig. 5.

The **signed weight** of a crossing on a band is defined according to Fig. 4. It depends on the direction of the half-twist of the band supporting the crossing.

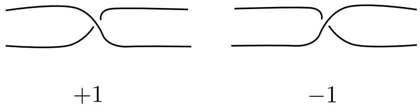

Fig. 4. The signed weight of a crossing on a band.

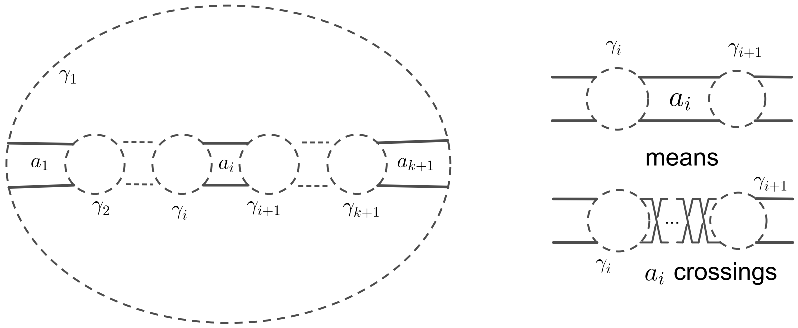

Fig. 5. A twisted band diagram.

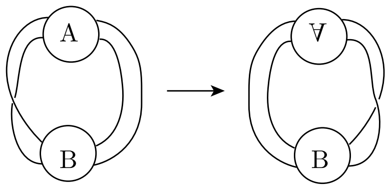

Fig. 6. A flype.

In Fig. 5 the boundary components of Σ are denoted $\gamma_1, \dots, \gamma_{k+1}$ where $k \geq 0$. The corresponding portion of the band between the circles γ_i and γ_{i+1} is called a **twist region** with $|a_i|$ crossing points. The sign of a_i is the signed weight of the $|a_i|$ crossing points. The integer a_i is called an intermediate weight. If $k+1 = 1$, the planar surface Σ is a disc and the twisted band diagram $(\Sigma, \Sigma \cap \Pi)$ is called a **spire** with $|a_1| \geq 2$ crossings. If $k+1 = 2$, the twisted band diagram is a **twisted annulus** and we require that $a_1 + a_2 \neq 0$.

A **flype** is a transformation of the projections of a knot as described by Fig. 6. The crossing in Fig. 6 which is moved by the flype is called an **active crossing**.

We ask the crossings on the same band diagram to have the same signed weight. In other words, using flypes and Reidemeister moves of type II, we can reduce the number of crossing points of a twisted band diagram of the projections of a given knot so that all non zero intermediate weights a_i of a twisted band diagram have the same sign.

Definition 2.5. (1) The crossings of a TBD (twisted band diagram) $(\Sigma, \Sigma \cap \Pi)$ are called the **visible crossings** of $(\Sigma, \Sigma \cap \Pi)$.
(2) The sum $a = \Sigma a_i$ is called the **total weight** of the twisted band diagram $(\Sigma, \Sigma \cap \Pi)$. If $k+1 \geq 3$ we may have $a = 0$. The absolute value of a is equal to the number of the visible crossings of $(\Sigma, \Sigma \cap \Pi)$.

Two Haseman circles are said to be **parallel** if they bound an annulus $A \subset \Sigma$ such that the pair $(A, A \cap \Gamma)$ is homeomorphic to Fig. 7.
We call a Haseman circle γ **boundary-parallel** if there exists an annulus $A \subset \Sigma$ such that:
(1) the boundary ∂A of A is the disjoint union of γ and a boundary component of Σ;
(2) $(A, A \cap \Gamma)$ is homeomorphic to Fig. 7.

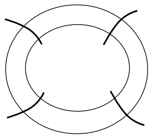

Fig. 7. Parallel Haseman circles.

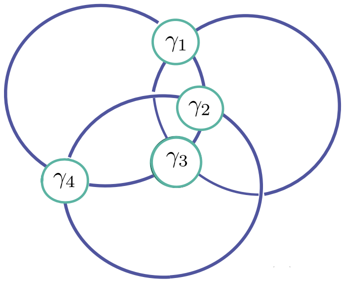

Fig. 8. A jewel.

Definition 2.6. A **jewel** is a diagram J such that:
(1) it is not a twisted band diagram with $k+1=2$ and $a=\pm 1$ or with $k+1=3$ and $a=0$.
(2) each Haseman circle of J is boundary-parallel.

Fig. 8 depicts a jewel $\mathrm{J}=(\Sigma, \Pi\cap\Sigma)$ where Σ is a planar surface with boundary $\partial\Sigma = \gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4$.

2.2. Families of Haseman circles for a projection

2.2.1. Canonical Conway circles

Unless otherwise stated, the projections we consider are connected and prime.

Definition 2.7. Let Π be a projection. A **family of Haseman circles** for Π is a set of Haseman circles satisfying the following conditions:
(1) any two circles are disjoint and
(2) no two circles are parallel.

Let $\mathcal{H}=\{\gamma_1,...,\gamma_n\}$ be a family of Haseman circles for Π. Let R be the closure of a connected component of $S^2\setminus\bigcup_{i=1}^{i=n}\gamma_i$. We call the pair $(R, R\cap\Pi)$ a diagram of Π determined by the family $\mathcal{H}$.

Definition 2.8. A family $\mathcal{C}$ of Haseman circles is an **admissible family** if each diagram determined by it is either a twisted band diagram or a jewel. An admissible family is **minimal** if removing a circle turns it into a family that is not admissible.

Theorem 2.1 is the main structure theorem about knot projections proved in ([17], Theorem 1). It is essentially due to Bonahon and Siebenmann.

Theorem 2.1 *(Existence and uniqueness theorem of minimal admissible families). Let* Π *be a knot projection in* S^2*. Then:*
i) there exist minimal admissible families for Π*;*
ii) if F_1 *and* F_2 *are two minimal admissible families for* Π*, then there is a homeomorphism* $h:(S^2,\Pi)\to(S^2,\Pi)$ *such that* $h(F_1)=F_2$*.*

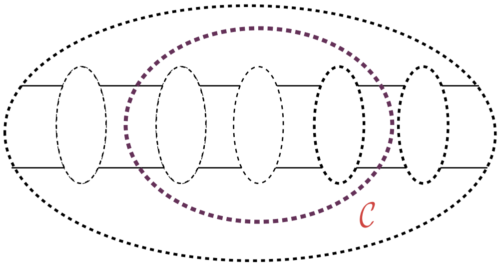

Fig. 9. A non canonical Conway circle.

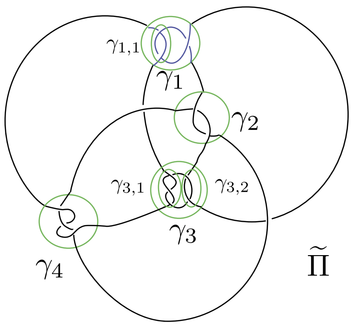

Fig. 10. Projection $\widetilde{\Pi}$ with its canonical Conway circles. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Definition 2.9. A Haseman circle belonging to "the" minimal admissible family of Π noted $\mathcal{C}_{can}$ is called a **canonical Conway circle** of the projection Π.

Example 1. The Haseman circle C in Fig. 9 is not a canonical Conway circle.

The decomposition of Π into twisted band diagrams and jewels determined by $\mathcal{C}_{can}$ will be called the **canonical decomposition** of Π. If there are no jewels in its canonical decomposition, the projection Π is said to be arborescent.

A canonical Conway circle can be of 3 types:
(1) a circle that separates two jewels,
(2) a circle that separates two twisted band diagrams,
(3) a circle that separates a jewel and a twisted band diagram.

Example 2. Fig. 10 illustrates a projection $\widetilde{\Pi}$ with its (green) canonical Conway family:

$$\mathcal{C}_{can}(\widetilde{\Pi}) = \{\gamma_1, \gamma_{1,1}, \gamma_2, \gamma_3, \gamma_{3,1}, \gamma_{3,2}, \gamma_4\}.$$

Remark 2.2.

1. As remarked in [8], our notion of jewel is more restrictive than the notion of John Conway polyhedron ([13] p. 139). We define a **jewel graph** G_J of a jewel J by collapsing each Haseman circle of J to a vertex. For Conway, the graph of a basic polyhedron is a simple regular graph of valency 4. A basic polyhedron can therefore be a tangle sum of several jewel graphs. A jewel graph is simply a polyhedron in the sense of Conway, indecomposable with regard to the tangle sum. The polyhedron 10^{***} has a non-trivial Haseman circle (see Fig. 11).

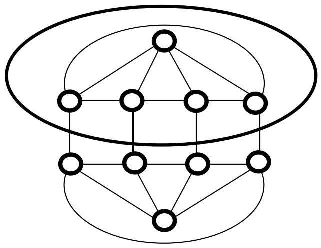

Fig. 11. 10^{***} is a tangle sum of two 6^*.

2. The minimal projection of the torus knot of type $(2, m)$ can be considered as a twisted band diagram with $k + 1 = 0$.

2.2.2. Essential Conway circles

Let Π be a projection on S^2.

Definition 2.10. A (2-dimensional) tangle $\mathcal{T}$ of Π is a pair $\mathcal{T} = (\Delta, \tau_\Delta)$ where Δ is a disk in the projection sphere S^2, τ_Δ is $\Pi \cap \Delta$ and the boundary $\partial\Delta$ of Δ intersects τ_Δ exactly on 4 no crossing points namely SE, NE, NW and SW which are located on $\partial\Delta$ in the south-east, north-east and so on. The **boundary** $\partial\mathcal{T}$ of $\mathcal{T}$ is the boundary $\partial\Delta$ of Δ.

Let $\mathcal{T} = (\Delta, \tau_\Delta)$ and $\mathcal{T}' = (\Delta, \tau\Delta')$ be two tangles with the same four endpoints SE, NE, NW and SW.

Definition 2.11. The tangles $\mathcal{T}$ and $\mathcal{T}'$ are **isotopic** if we can change $\mathcal{T}$ into $\mathcal{T}'$ by a sequence of Reidemeister moves inside Δ, keeping the four endpoints SE, NE, NW and SW fixed.

Definition 2.12. A **rational tangle** is a tangle such that all its canonical Conway circles are concentric and delimit twisted annuli, with the exception of the innermost circle which is the boundary of a spire, as shown in Fig. 12.
A **maximal rational tangle** of a knot projection Π is a rational tangle that is not strictly included in a larger rational tangle of Π.

Let $\mathcal{T}$ be a rational tangle. We now consider $\mathcal{T}$ under the **Cardan** form $\mathrm{T}[a_0, \ldots, a_m]$ (or equivalently under the standard form described in [12]) illustrated in Fig. 12 such that the twisted band diagrams have weights $b_i = (-1)^i a_i$ with $i = 0, \ldots, m$ and such that the first weight band b_0 is horizontal.

To $\mathrm{T}[a_0, \ldots, a_m]$ where $a_0 \in \mathbb{Z}$ and $a_1, \ldots, a_m \in \mathbb{Z} - \{0\}$, we assign the continued fraction

$$[a_0, a_2, \cdots a_m] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_m}}}$$

If $\mathcal{T}$ is not the trivial tangle $\mathrm{T}([\infty])$ (Fig. 3(a)), the rational number $\frac{r}{s} = [a_0, a_2, \cdots a_m]$ with $(r, s) = 1$ and $r > 0$ is called the **fraction** $\mathrm{F}(\mathcal{T})$. By convention, the fraction of the trivial tangle $\mathrm{T}([\infty])$ is: $\mathrm{F}(\mathrm{T}([\infty]) := \infty$.

The fraction is an isotopy invariant of the tangle $\mathcal{T}$. It means that with the expansion of $\frac{r}{s}$ in another continuous fraction $[d_0, \ldots, d_k]$, we get another Cardan tangle $\mathrm{T}[d_0, \ldots, d_k]$ isotopic to $\mathrm{T}[a_0, \ldots, a_m]$. We will use $\mathrm{T}_{\frac{r}{s}}$ to denote a rational tangle with fraction $\frac{r}{s}$.

Remark 2.3. Let $\frac{r}{s}$ be a rational number with $r > 0$ and $(r, s) = 1$. Then $\frac{r}{s}$ has an expansion $[a_0, \ldots, a_m]$ where the a_i's are all positive or all negative; it is called a homogeneous continued fraction. If $[a_0, \ldots, a_m]$ is a homogeneous continued fraction, the Cardan tangle $\mathrm{T}[a_0, \ldots, a_m]$ is an alternating tangle.

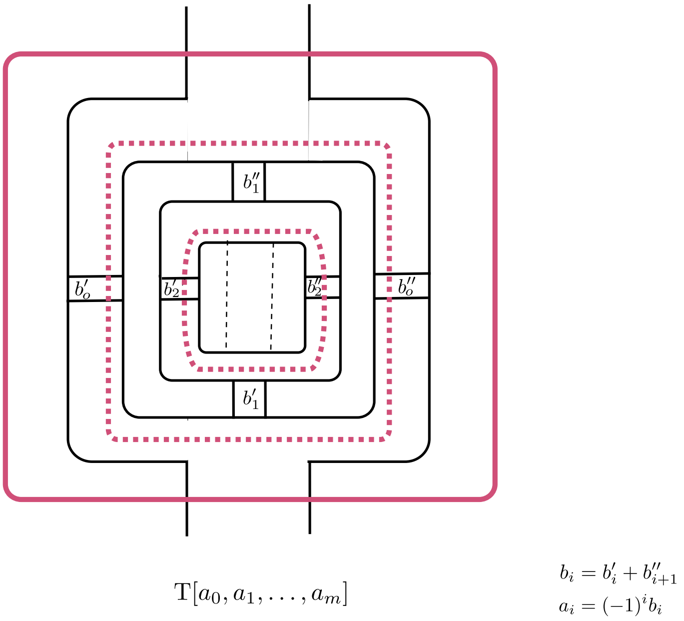

Fig. 12. Cardan tangle $\mathrm{T}[a_0, \ldots, a_m]$.

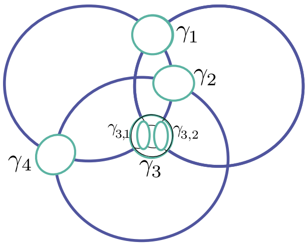

Fig. 13. The essential Conway family of $\widetilde{\Pi}$.

Definition 2.13. An **essential Conway circle** of an alternating projection Π is a canonical Conway circle that is not properly contained in a maximal rational tangle.

In a rational knot projection, there are no essential Conway circles.

Let Π be a non-rational knot projection. By removing from the minimal admissible family $\mathcal{C}_{can}$ of Π all concentric Conway circles of each maximal rational tangle $\mathcal{T}$ of (S^2, Π) except its boundary circle $\partial\mathcal{T}$, we obtain the **essential Conway family** of Π denoted $\mathcal{C}_{ess}(\Pi)$.

Remark 2.4. The set of essential Conway circles is empty for an alternating projection if and only if the projection is one of the three following cases:
a) a standard torus knot projection of type $(2, s)$ (in this case it can be considered as a twisted band diagram with empty boundary (Remark 2.2)),
b) a jewel without boundary,
c) a minimal projection of a rational knot.

Example 3. 1) Fig. 13 illustrates the essential Conway family $\mathcal{C}_{ess}(\widetilde{\Pi})$ of the projection $\widetilde{\Pi}$ of Fig. 10:

$$\mathcal{C}_{ess}(\widetilde{\Pi}) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_{3,1}, \gamma_{3,2}, \gamma_4\}.$$

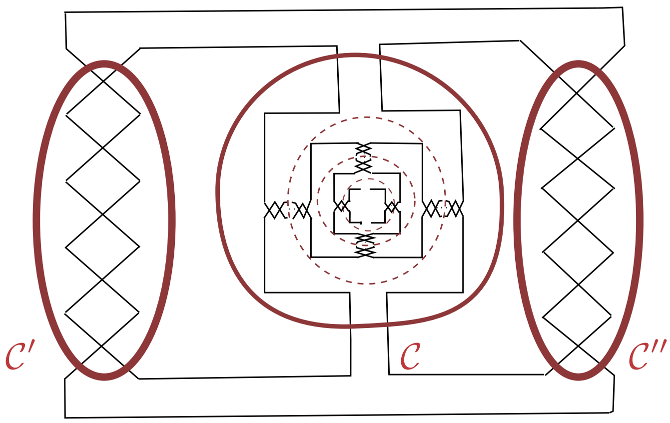

Fig. 14. A projection with its set $\mathcal{C}_{ess} = \{\mathcal{C}, \mathcal{C}', \mathcal{C}''\}$.

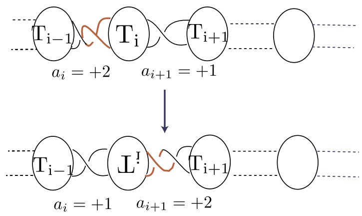

Fig. 15. An efficient flype.

2) In the projection depicted in Fig. 14, the circles $\mathcal{C}'$, $\mathcal{C}''$ and $\mathcal{C} = \partial\mathrm{T}[a_0, \ldots, a_m]$ are essential Conway circles, while dotted red circles are only canonical Conway circles.

2.2.3. Position of flypes with respect to the canonical Conway decomposition

We can now precisely locate where flypes can be performed with respect to the canonical Conway decomposition of a prime alternating reduced knot.

Theorem 2.2. *[17] (Position of flypes) Let Π be a prime alternating reduced knot projection in S^2 and suppose that a flype can be done in Π. Then its active crossing point belongs to a twisted band diagram determined by $\mathcal{C}_{can}(\Pi)$. The flype moves the active crossing point either within the twist region to which it belongs or to another twist region of the same twisted band diagram.*

Remark 2.5. (1) We are only interested in **efficient** flypes that move the active crossing point from one twist region to another in the same twist band diagram (see Fig. 15).

(2) Let $\mathcal{T}$ be a TBD of Π and $C_1, C_2, \ldots, C_k, C_1$ be its cyclic sequence of $\mathcal{C}_{can}|_{\mathcal{T}}$. A flype on $\mathcal{T}$ does not modify the order of occurrence of the Conway circles $\mathcal{C}_{can}|_{\mathcal{T}}$ in the cyclic sequence.

Definition 2.14. (1) The set of the twist regions of a given twisted band diagram is called a **flype orbit** (Fig. 16).
(2) The cardinal of $\mathcal{C}_{can}|_{\mathcal{T}}$ is the **valency** $k + 1$ of the TBD $\mathcal{T}$.

Corollary 2.1. *[17]*
(1) A flype moves an active crossing point inside the flype orbit to which it belongs.
(2) Two distinct flype orbits are disjoint.

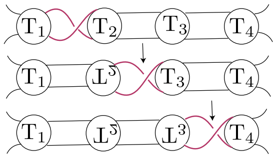

Fig. 16. A flype orbit.

This implies that an active crossing point belongs to one and only one TBD. Since two TBDs share at most one canonical Conway circle, Corollary 2.1 can be interpreted as a loose kind of commutativity of flypes.

2.3. Canonical and essential structure trees

We now focus on the canonical decomposition of alternating knot projections.

2.3.1. The flyping theorem

Fundamental to our purpose is the Menasco-Thistlethwaite's Flyping Theorem ([15]). It is necessary to make the notion of flype precise.

Let Π be a n-crossing regular projection of a link L on the projection sphere S^2. As in [15], consider n disjoint small "crossing ball" neighborhoods $B_1, \ldots, B_n$ of the crossing points $c_1, \ldots c_n$ of Π. Then, assume that L coincides with Π, except that inside each B_i the two arcs forming $(c_i) = \Pi \cap B_i$ are perturbed vertically to form semicircular overcrossing and undercrossing arcs $\alpha(c_i)$, which lie on the boundary of B_i. This relationship between the link L and its projection Π is expressed as $L = \lambda(\Pi)$ ([15]). Note that there is a homeomorphism of pairs $(S^3, L) \to (S^3, \lambda(\Pi))$. We call $\lambda(\Pi)$ a **realized projection** (or a realized diagram in the terms of [2]) for L.

We can consider the ambient space S^3 to be $\mathbb{R}^3 \cup \{\infty\}$, and we shall take the 2-sphere S^2 on which the regular projection Π lies to be $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\}$. We assume that the knot $\lambda(\Pi)$ lies within the neighborhood $N = \{x \in \mathbb{R}^3 : \frac{1}{2} \leq \|\mathbf{x}\| \leq \frac{3}{2}\}$.

Definition 2.15. Let $g : (S^3, \lambda(\Pi_1)) \to (S^3, \lambda(\Pi_2))$ be a homeomorphism of pairs. The homeomorphism of pairs g is **flat** if g is pairwise isotopic to a homeomorphism of pairs h with the condition that h maps N onto itself and $h|_N = h_0 \times id_{[\frac{1}{2},\frac{3}{2}]}$ for some orientation-preserving homeomorphism $h_0 : S^2 \to S^2$. We call h_0 the **principal part of the flat homeomorphism** g.

Definition 2.16. An **isomorphism of realized projections** $h : \lambda(\Pi) \to \lambda(\Pi_*)$ is a homeomorphism of pairs $h : (S^3, L) \to (S^3, \widetilde{L})$ such that:
(1) $h(S^2) = S^2$,
(2) $h(B_i) = \widetilde{B}_i$, and
(3) $h(\alpha(c_i)) = \alpha(\widetilde{c}_i)$.

Given a flat homeomorphism g and asking the crossing balls to be sent on the crossing balls, g is an isomorphism of realized projections.

We recall the definition of a flype as described in [15].

Definition 2.17. Let Π_1 be a projection with the pattern described in Fig. 6(a). A **standard flype** of $(S^3, \lambda(\Pi_1))$ is any homeomorphism f which maps $(S^3, \lambda(\Pi_1))$ to a pair $(S^3, \lambda(\Pi_2))$ where Π_2 is the pattern described in Fig. 6(b), in such a way that:
(1) f sends the 3-ball B_{A} into itself by a rigid rotation about an axis in the projection plane,

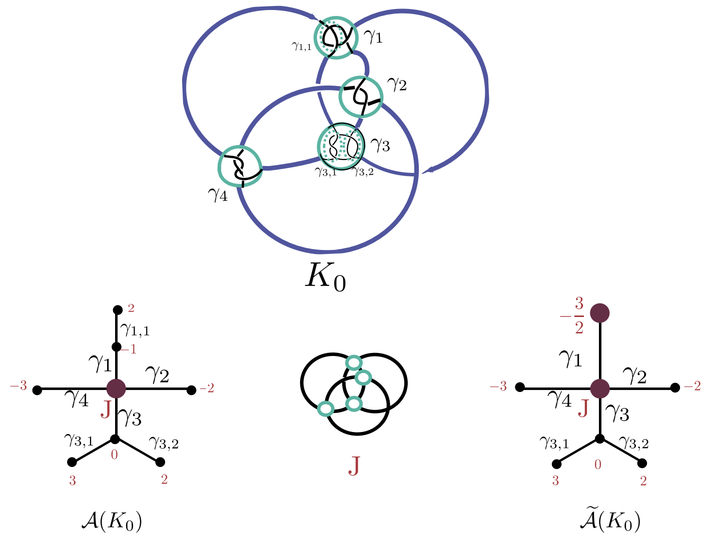

Fig. 17. Canonical structure tree and essential structure tree of K_0.

(2) f fixes pointwise the 3-ball B_{B},
(3) f moves the crossing visible on the left of Fig. 6(a) to the crossing visible on the right of Fig. 6(b).

Definition 2.18. Let Π_1 be any projection. A flype is any homeomorphism $f : (S^3, \lambda(\Pi_1)) \to (S^3, \lambda(\Pi_1))$ of the form $f = g_1 \circ f' \circ g_2$ where f' is a standard flype and g_1 and g_2 are flat homeomorphisms.

If the tangle A of Fig. 6 contains no crossing, then the standard flype determined by that figure is a flat homeomorphism; therefore according to the above definition, any flat homeomorphism is a flype. We call it a **trivial flype**.

Let us state Menasco-Thistlethwaite's Flyping Theorem:

Theorem 2.3 *([15]). Let Π_1 and Π_2 be two reduced, prime, oriented, alternating projections of knots. If $f : (S^3, \lambda(\Pi_1)) \to (S^3, \lambda(\Pi_2))$ is a homeomorphism of pairs, then f is a composition of flypes and flat homeomorphisms.*

2.3.2. Canonical structure tree $\mathcal{A}(K)$

Since two realizations of alternating projections in S^2 of the same isotopy class of an oriented prime alternating knot in S^3 are related by flypes, their canonical and essential **structure trees** constructed as described below, are isomorphic.

Construction of the canonical structure tree $\mathcal{A}(K)$ Let K be a prime alternating knot and let Π be an alternating projection of K. Let $\mathcal{C}_{can}$ be the canonical Conway family for Π. We construct the canonical structure tree $\mathcal{A}(K)$ as follows: its vertices are in bijection with the diagrams determined by $\mathcal{C}_{can}$ and its edges are in bijection with the canonical circles; the vertices of each edge γ represent the diagrams having the canonical circle γ in their boundary. Since S^2 has Euler characteristic 2, the constructed graph is a tree.

We label the vertices of $\mathcal{A}(K)$ as follows: if a vertex represents a twisted band diagram, we label it by its total weight a and if it represents a jewel, we label it with the letter J.

Example 4. The knot K_0 with the projection $\widetilde{\Pi}$ represented by Fig. 10 has its canonical structure tree $\mathcal{A}(K_0)$ given by Fig. 17.

In the case of a tangle $\mathcal{T}$ whose boundary is a canonical Conway circle γ, the canonical structure tree $\mathcal{A}(\mathcal{T})$ of $\mathcal{T}$ is a graph such that all its edges have two vertices at the extremities except for one “open” edge (with a single vertex) which represents the circle γ. For an example, see Fig. 18.

Proposition 2.1. *The canonical structure tree $\mathcal{A}(K)$ is independent of the alternating projection chosen to represent K.*

Proof. Let Π be an alternating knot projection in S^2. By Theorem 2.1 there exist minimal Conway families for Π and by Theorem 2.2 a flype or a flat homeomorphism does not modify canonical structure tree. By Theorem 2.3, we conclude that the canonical structure tree is independent of the chosen alternating projection Π and we can speak of the canonical structure tree of K (and not only of Π). □

Definition 2.19. The alternating knot K is arborescent if each vertex of $\mathcal{A}(K)$ has an integer weight.

Remark 2.6. If the projection Π is arborescent, we can encode Π with a **weighted planar tree** à la Bonahon-Siebenmann (§5 in [17]), which is a canonical structure tree with more complete information.

2.3.3. Essential structure tree $\widetilde{\mathcal{A}}(K)$

Construction of the essential structure tree $\widetilde{\mathcal{A}}(L)$ On the same lines of the construction of the canonical structure tree $\mathcal{A}(K)$, we construct the **essential structure tree** $\widetilde{\mathcal{A}}(K)$. The vertices of $\widetilde{\mathcal{A}}(K)$ are in bijection with the diagrams determined by the set $\mathcal{C}_{ess}(\Pi)$ and the edges are in bijection with the circles of $\mathcal{C}_{ess}(\Pi)$. Two vertices corresponding to two diagrams determined by $\mathcal{C}_{ess}(\Pi)$ are joined by an edge of $\widetilde{\mathcal{A}}(K)$ if they share a common essential circle in their boundary.

As in the case with the canonical structure tree, Flyping Theorem implies that:

Proposition 2.2. *The essential structure tree $\widetilde{\mathcal{A}}(K)$ is independent of the minimal projection chosen to represent K.*

The essential structure tree of a tangle: To a tangle $\mathcal{T}$ with an essential Conway circle γ as boundary, we associate an essential structure tree denoted $\widetilde{\mathcal{A}}(\mathcal{T})$ which has only one “open edge” with one vertex-end. The unique edge of $\widetilde{\mathcal{A}}(\mathcal{T})$ corresponds to γ (see an example in Fig. 18).

Remark 2.7.

1. If $\mathrm{T}_{\frac{p}{q}}$ is a maximal rational tangle, $\widetilde{\mathcal{A}}(\mathrm{T}_{\frac{p}{q}})$ is a linear graph consisting only of an “open” edge and a vertex labeled by $\frac{p}{q}$.
2. A vertex in $\widetilde{\mathcal{A}}(K)$ with weight $\in \mathbb{Q} \setminus \mathbb{Z}$ is monovalent and its union with its single edge corresponds to a maximal rational tangle in Π.
3. Only monovalent vertices of an essential structure tree of a knot can have weights that are $\in \mathbb{Q} \setminus \mathbb{Z}$

The essential structure tree $\widetilde{\mathcal{A}}(\mathrm{T}_{\frac{r}{s}})$ is reduced to a vertex of weight $\frac{r}{s}$.

Example 5. $\mathcal{A}(\mathrm{T}[-1,-2])$ and $\widetilde{\mathcal{A}}(\mathrm{T}[-1,-2])$ are described in Fig. 18.

Remark 2.8. The essential structure tree $\widetilde{\mathcal{A}}(K)$ is reduced to a single vertex if and only if K is a rational knot $\mathrm{T}_{\frac{r}{s}}$ or a knot described by a jewel without boundary.

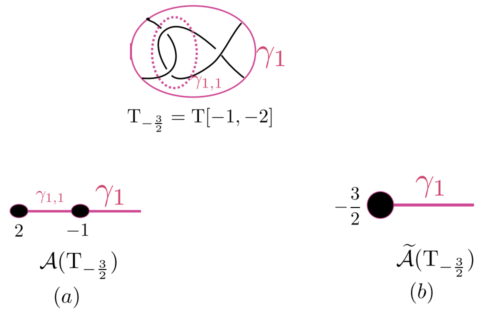

Fig. 18. (a) Canonical structure tree $\mathcal{A}(\mathrm{T}[-1,-2])$ and (b) Essential structure tree $\widetilde{\mathcal{A}}(\mathrm{T}[-1,-2])$.

3. On Visibility Theorem 3.1

This section is about the proof of the Visibility Theorem 3.1 for q-periodic alternating prime knots:

Theorem 3.1. *Let K be an oriented prime alternating knot that is q-periodic with $q \geq 3$. Then there exists a q-periodic alternating projection $\widetilde{\Pi}$ for K.*

We first recall the definition of a q-periodic knot in S^3.

Definition 3.1. A knot K is q-periodic if there is a (auto)-homeomorphism Φ of pairs (S^3, K) which satisfies the following conditions:
(1) Φ is a $\frac{2\pi}{q}$-rotation about a "line" (circle) α in S^3 and
(2) $\alpha \cap K = \varnothing$.
Φ is called a q-periodicity of (S^3, K).

In the case $q > 2$, the condition (2) is a consequence of $\Phi(K) = K$.

We now define the notion of visibility of a q-periodicity of an alternating knot.

Definition 3.2. Let K be an alternating q-periodic knot. The q-periodicity Φ of K is **visible** if there exists an alternating projection (S^2, Π) of K and a homeomorphism of pairs $h : (S^3, K) \to (S^3, \lambda(\Pi))$, where $\lambda(\Pi)$ is a realized projection of Π such that $h \circ \Phi \circ h^{-1} : (S^3, \lambda(\Pi)) \to (S^3, \lambda(\Pi))$ and its restriction to S^2 is a $\frac{2\pi}{q}$-rotation. The projection Π is called a **q-periodic projection**.

Remark 3.1. Let K be a q-periodic knot with Φ a q-periodicity of (S^3, K). For each divisor p of q, K is p-periodic with its p-periodicity $\Psi_r = \Phi^r$ where $r = \frac{q}{p}$.

For our goal, we introduce the following notion:

Definition 3.3. A knot K is **strictly q-periodic** if K is q-periodic but not rq-periodic for any $r \geq 2$.

A projection Π is strictly q-periodic if it is q-periodic but not rq-periodic for any $r \geq 2$.

Example 6. Let Π be an alternating projection described in Fig. 20 where Y is an alternating tangle with boundary an essential Conway circle. The big red circles do not belong to $\mathcal{C}_{ess}(\Pi)$, and they do not appear on $\widetilde{\mathcal{A}}(K)$. The projection Π is not strictly 4-periodic since as shown by Fig. 19, Π is an 8-periodic projection. Note that the projection Π depicted in Fig. 20 is not a knot projection, whatever the tangle Y (see Proposition 3.3 below).

From now on, by q-periodic projections, we mean strictly q-periodic projections.

Our objective being the study of periodicity, we reformulate the Flyping Theorem in the following form:

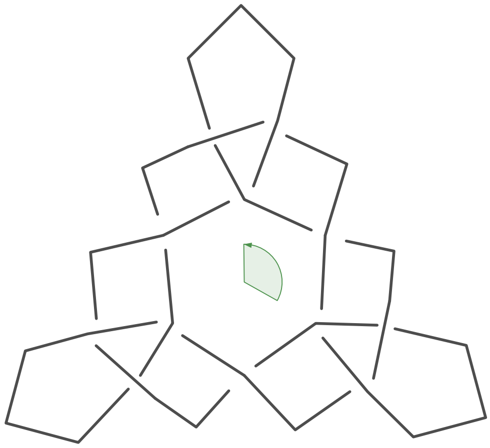

Fig. 19. An example of 3-periodic projection.

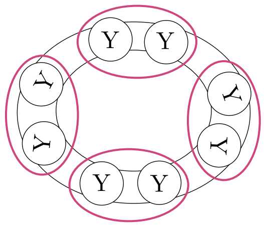

Fig. 20. A non-strictly 4-periodic projection.

Theorem 3.2. *Let $\Phi : (S^3, K) \to (S^3, K)$ be an orientation preserving homeomorphism of pairs where K is a prime alternating knot. Let $\lambda(\Pi)$ be a realized projection of a reduced alternating projection Π of K and $h : (S^3, K) \to (S^3, \lambda(\Pi))$ be a homeomorphism of pairs. Then the isomorphism of realized projections $\Phi_\Pi = h \circ \Phi \circ h^{-1} : (S^3, \lambda(\Pi)) \to (S^3, \lambda(\Pi))$ can be expressed as $\Phi_\Pi = \phi \circ F$ where ϕ is a flat homeomorphism and F is a composition of standard flypes on $\lambda(\Pi)$ unless F is the identity.*

Remark 3.2.

1. For our purpose, we separate the flat homeomorphisms from the flypes. Therefore, the standard flypes involved in Flyping Theorem 3.2 above are not trivial.
2. Standard flypes and flat homeomorphisms "essentially" commute. By "essentially" commute, we mean that if f is a standard flype and h is a flat homeomorphism, then $f \circ h = h \circ f'$ where $f' = h^{-1} \circ f \circ h$ is also a standard flype. Then it is possible to express $\Phi_\Pi = \phi \circ F$.

Definition 3.4. An **essential Conway sphere** of Π is a 2-dimensional sphere lying in the interior of $N = \{x \in \mathbb{R}^3 : \frac{1}{2} \le \|\mathbf{x}\| \le \frac{3}{2}\}$ such that its projection on the projection plane is an essential Conway circle.

Consider the set $\mathcal{C}_{ess}(\Pi)$ of essential Conway circles of a knot projection Π and its corresponding set $\mathcal{S}_{ess}(\Pi)$ of essential Conway spheres.

Definition 3.5. A **3-dimensional tangle** is a pair (B, t), where B is a 3-ball and t is a proper 1-submanifold of B meeting ∂B in four fixed points, namely SE, NE, NW and SW on the equatorial circle of ∂B.
Two oriented tangles (B, t_1) and (B, t_2) are equivalent if there is an orientation-preserving auto-homeomorphism $\phi : B \to B$ such that ϕ keeps ∂B fixed and $\phi(t_1) = \phi(t_2)$.

A 2-dimensional tangle in Definition 2.10 is the projection of a 3-dimensional tangle on the equatorial disk of the 3-ball B which contains the four points SE, NE, NW and SW.

From a (2-dimensional) tangle $\mathcal{T}$, we can create a 3-dimensional tangle $\lambda(\mathcal{T})$ by means of a suitably small vertical perturbation near each crossing of the diagram; the ambient space of $\lambda(\mathcal{T})$ is considered as a 3-ball for which the disk region of $\mathcal{T}$ is an equatorial slice.

3.1. First steps of the proof of the Visibility Theorem

Let $K \subset S^3$ be a prime (strictly) q-periodic alternating knot with $\Phi : (S^3, K) \to (S^3, K)$ its corresponding rotational symmetry of order q. Let Π be a reduced alternating projection of K and $\lambda(\Pi)$ its realized diagram. By Theorem 3.2, Φ is conjugate through maps of pairs to an isomorphism Φ_Π on $\lambda(\Pi)$ (onto itself) which is a composition of a flat homeomorphism with standard flypes: $\Phi_\Pi = \phi \circ F$.

We have two cases:

(1) Suppose no flypes are needed. Hence $\Phi_\Pi = \phi$ is flat and $\phi^q = \mathrm{Id}$. By Kerékjárto's theorem ([3]), the principal part of ϕ which is a homeomorphism of (S^2, Π) is topologically conjugate to a rotation of S^2 of order q without fixed points on Π. Consequently, the q-periodicity of K is visible on an alternating projection Π of K.

Remark 3.3.

1. If (S^2, Π) is a q-periodic jewel without boundary, its q-periodicity is visible on Π because the jewels have no TBD and then no flypes are necessary to realize Φ_Π.
2. A torus knot of type $(2, q)$ (q must therefore be odd) displays the q-periodicity on a standard alternating projection.

(2) In what follows, we will deal with the case where flypes may be involved.

In §3.2, we will describe how Φ_Π acts on the structure trees and on the set of essential Conway spheres.

3.2. Action of Φ_Π on structure trees

Let K be an alternating q-periodic knot and $\Phi : (S^3, K) \to (S^3, K)$ be the q-periodicity. Let $(S^3, \lambda(\Pi))$ be a realized alternating projection and $\Phi_\Pi : (S^3, \lambda(\Pi)) \to (S^3, \lambda(\Pi))$ be the homeomorphism of pairs given by Φ.

Proposition 3.1. *Φ_Π induces a permutation σ_{Φ_Π} on the set of essential Conway spheres $\mathcal{S}_{ess}(\Pi)$ such that $\sigma^q_{\Phi_\Pi}$ is the identity permutation.*

Proof. Each essential sphere $S \in \mathcal{S}_{ess}(\lambda(\Pi))$ intersects the projection sphere S^2 in an essential circle of Π. By the Theorem 2.1 $\Phi_\Pi(S \cap S^2)$ is isotopic to an essential Conway circle $S' \cup S^2$ where $S' \in \mathcal{S}_{ess}(\lambda(\Pi))$. Then we define $\sigma_{\Phi_\Pi}(S) = S'$. Since Φ has order q then $\sigma^q_{\Phi_\Pi}$ is the identity permutation. □

Corollary 3.1. *Φ_Π induces an automorphism $\widetilde{\Phi}$ on the tree $\widetilde{\mathcal{A}}(K)$. The automorphism $\widetilde{\Phi}$ on $\widetilde{\mathcal{A}}(K)$ satisfies $\widetilde{\Phi}^q = \mathrm{Identity}$.*

Proof. The automorphism $\widetilde{\Phi}$ is determined completely by the permutation σ_{Φ_Π} given by Proposition 3.1, □

Since the graph $\widetilde{\mathcal{A}}(K)$ is a tree, the fixed point set $Fix(\widetilde{\Phi})$ is a non-empty subtree. So we have two possibilities:

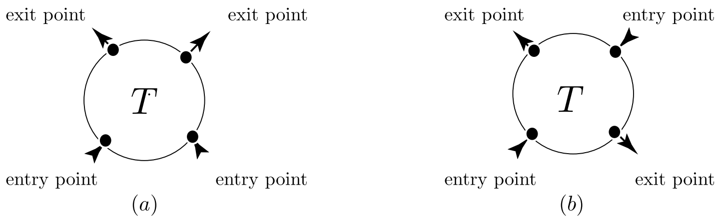

Fig. 21. Orientation of the boundary points.

1. $Fix(\widetilde{\Phi})$ contains an edge E of $\widetilde{\mathcal{A}}(K)$ (a Conway circle is invariant by Φ_Π).
2. $Fix(\widetilde{\Phi})$ is reduced to a vertex V_0 of $\widetilde{\mathcal{A}}(K)$.

Remark 3.4. If $\widetilde{\mathcal{A}}(K)$ is reduced to a single vertex V_0 then K is either a rational knot or a knot corresponding to a jewel without boundary and the automorphism $\widetilde{\Phi}$ is obviously the identity map.

3.2.1. Analysis of $Fix(\widetilde{\Phi})$

In order to describe the two cases of $Fix(\widetilde{\Phi})$ stated in §3.1 in terms of the essential decomposition of (S^2, Π), let us first describe how the boundary points of a tangle are oriented.

Let $\mathcal{T}$ be a tangle of a projection Π. The intersection points of $\partial\mathcal{T} \cap \Pi$ are called the **boundary points** of $\mathcal{T}$. Consider an orientation on K, by the orientation and the connectedness of Π, the four boundary points of $\mathcal{T}$ are oriented so that two are entry points and the other two are exit points (see Fig. 21). Up to a change in the global orientation of the strands and up a rotation of angle $\frac{\pi}{2}$, we have the two possible configurations described in Fig. 21.

Proposition 3.2. *Let γ be a canonical or essential Conway circle of Π. If γ is Φ_Π-invariant then $q = 2$.*

Proof. Since Φ_Π is topologically conjugate to a $2\pi/q$-rotation, except the fixed points, the orbits of each point under the action of Φ_Π on S^3 have q points. Since Φ_Π preserves $\gamma \cap K$ and there are no fixed points by the action of Φ_Π on K, we have that the set $\gamma \cap K$ is either an orbit of Φ_Π with $q = 4$ points, or two orbits with each $q = 2$ points.

Assume $q = 4$. Let Δ_1 and Δ_2 be the two disks in the projection sphere such that γ is $\partial\Delta_1 = \partial\Delta_2$. The two disks Δ_1 and Δ_2 are either permuted or invariant by Φ_Π.

Assume Φ_Π permutes the two disks Δ_1 and Δ_2. Let S_γ be a Conway sphere for $\lambda(\Pi)$ intersecting to the projection sphere at γ and which is invariant by Φ_Π. The homeomorphism $\Phi_\Pi|^2_{S_\gamma}$ is of order two and preserves the orientation of S_γ. Therefore, by Kerékjárto Theorem, it is topologically conjugate to a rotation of order 2 of S_γ. Thus, $\Phi_\Pi|^2_{S_\gamma}$ has two fixed points, which must also be the fixed points of $\Phi_\Pi|_{S_\gamma}$. This implies that $\Phi_\Pi|_{S_\gamma}$ preserves the orientation and that it is topologically conjugate to a rotation. Then the two connected components of $S^3 - S_\gamma$ are invariant by Φ_Π, but this contradicts the hypothesis that $\Phi_\Pi(\Delta_1) = \Delta_2$. So the two disks Δ_1 and Δ_2 are invariant by Φ_Π.

Since $\Phi_\Pi(\Delta_i) = \Delta_i$ and since $\gamma \cap K$ is a orbit of the action of Φ_Π, the points in $\gamma \cap \Pi$ would be all entry points or all exit points and that is impossible. Therefore $q = 2$. □

Corollary 3.2. *If $q \geq 3$, there are no edges in $Fix(\widetilde{\Phi_\Pi})$ and $Fix(\widetilde{\Phi_\Pi}) = V_0$ where V_0 is a vertex of $\widetilde{\mathcal{A}}(K)$.*

3.3. Proof of Visibility Theorem 3.1 and applications

3.3.1. Proof of Visibility Theorem 3.1

According to Remark 2.8, if the q-periodic knot K is a jewel without boundary or a torus knot of type $(2, q)$, we are done.

Since the non-torus rational knots are only 2-periodic (see for instance Theorem 3.1 in [10]), the hypothesis $q \geq 3$ excludes the case of rational knots.

There remains the case of a projection Π whose $\mathcal{C}_{ess}(\Pi)$ is not empty. According to Corollary 3.2, the set $Fix(\widetilde{\Phi_\Pi})$ is reduced to a vertex V_0 representing a jewel or a TBD.

Proof of the Visibility Theorem. We will denote V_0 the vertex in the essential structure tree such that $Fix(\widetilde{\Phi_\Pi}) = \{V_0\}$.

Case 1. The vertex V_0 corresponds to a jewel $J_0 = (\Sigma, \Sigma \cap \Pi)$ with non-empty boundary.

Let $\gamma_1, \dots, \gamma_k$ be the boundary components of Σ. Each essential Conway circle γ_i bounds on S^2 a disk Δ_i which does not meet the interior of J_0. Consider the tangles $\mathcal{T}_i = (\Delta_i, \tau_{\Delta_i})$ where $i = 1, \dots, k$. Note that the k underlying discs are distinct. Since J_0 is a jewel, no flypes can occur in J_0. Since $\widetilde{\Phi_\Pi}$ does not leave any edge of the essential tree invariant, there is no invariant boundary of Σ by $\Phi_\Pi|_\Sigma$. By Kerékjárto's theorem applied to $\Phi_\Pi|_\Sigma$ we have that $\Phi_\Pi|_\Sigma$ is topologically conjugate to a rotation of S^2 with the two fixed points in the interior of Σ.

We can modify Π such that $\Phi_\Pi|_\Sigma$ is a rotation of order q which acts freely on the k boundary components of Σ. We can also modify Π and Σ so that the boundary components of Σ are circles. After these modifications, we continue to denote the new projection and homeomorphism respectively by Π and Φ_Π.

Each circle γ_i has q images in its orbit by Φ_Π. Thus, $k = nq$ and we have k distinct tangles $\mathcal{T}_i = (\Delta_i, \Pi \cap \Delta_i)$ with underlying disks Δ_i where $i = 1, \dots k$. Note that the k boundary components of Σ correspond to the k adjacent vertices to V_0. Consider a set of representatives $\{\Delta_{i_j}\}_{j=1,..,n}$ of the orbits of Φ_Π on the set of disks $\{\Delta_i\}_{i=1,..,k}$.

Let $\mathcal{T}$ be a TBD in Δ_{i_j} and let t be a twist region in $\mathcal{T}$. We modify $\mathcal{T}$ by flypes so that all visible crossings of $\mathcal{T}$ move to the twist region t. Similarly, using flypes, we modify $\Phi_\Pi^j(\mathcal{T})$ such that all its visible crossings are in the twist region $\Phi_\Pi^j(t)$, $j = 1, ..., q-1$. We perform the same operations on all the TBDs of Δ_{i_j} and in all the orbits in the set of discs $\{\Delta_i\}_{i=1,..,k}$. Let Π' be the new projection of K resulting from the process described above. Note that Π and Π' differ only in certain twist regions of TBDs, then we define $\Phi'_{\Pi'}$ equal to Φ_Π outside these twist regions and as a flat homeomorphism on the TBDs.

By the Flyping Theorem we can express $\Phi'_{\Pi'} = \phi \circ F$, where ϕ is a flat homeomorphism. Since $\Phi'_{\Pi'}$ contains no more flypes in the TBDs $\Phi'_{\Pi'} = \phi$ is flat. By Kerékjárto's theorem $\Phi'_{\Pi'}$ is conjugate to a q-rotation whose two fixed points are inside J_0.

Case 2. The vertex V_0 corresponds to a TBD $\mathcal{T}_0 = (\Sigma, \Sigma \cap \Pi)$.

Since $\mathcal{T}_0$ is invariant by Φ_Π, the number of the visible crossings of $\Sigma \cap \Pi$ is mq. Since there are no edges invariant by $\widetilde{\Phi}_\Pi$, the boundary components of Σ are partitioned into Φ_Π-orbits of q elements and the total number of these components is sq for some integer $s \geq 1$. By Kerékjárto's theorem, ϕ is equivalent to a rotation of order q with the two fixed points in Σ. We first modify the projection Π and Σ, if necessary, such that the TBD $\mathcal{T}_0 = (\Sigma, \Sigma \cap \Pi)$ becomes the symmetrical TBD $\mathcal{T}'_0 = (\Sigma', \Sigma' \cap \Pi')$ where Σ' is invariant by a q-rotation R of the sphere S^2:

- the visible crossings of $\mathcal{T}'_0$ are in q twist regions, such that each twist region has m crossings and these twist regions are symmetric with respect to the rotation R and
- the boundary components of Σ' are sq circles partitioned into s orbits of the action of R.

See an example in Fig. 22, where $q = 4$, $m = 1$ and $s = 2$.

We call again Π and Φ_Π the resulting projection and homeomorphism. By a process, similar to that described in Case 1, we consider the discs Δ_i in the projection sphere such that $\Sigma \cap \Delta_i$ is a boundary

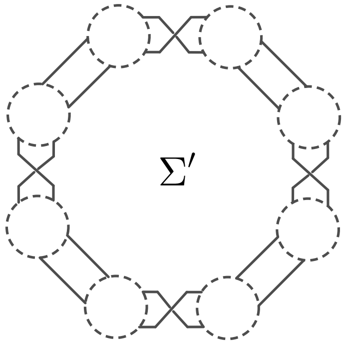

Fig. 22. Example of the symmetrical modified TBD corresponding to V_0 in Case 2.

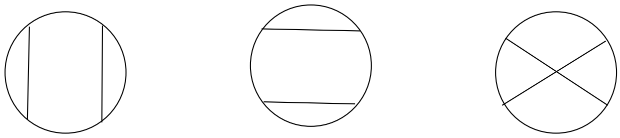

Fig. 23. Connection paths of a tangle.

component of Σ. In each orbit of discs Δ_i by the action of Φ_Π we will perform flypes if necessary inside the TBDs in order to modify Π and Φ_Π so that Φ_Π restricted to such tangles is flat. Proceeding in this way in all the orbits, we obtain a projection Π' visualizing the symmetry. □

Remark 3.5. The proof of the Theorem 3.1 gives a method to find out if an alternating knot is q-periodic from an alternating projection. First, we construct the essential graph and find the possible automorphisms of order q. Then, we study if such an automorphism of the graph can be realized as the automorphism given by the q-periodicity of an alternating projection and this last step is a finite process.

Question: Are there any restrictions on the values of q in Visibility Theorem 3.1?

Proposition 3.3. *For prime alternating* **knots** *where* $Fix(\widetilde{\Phi}) = V_0$ *with* V_0 *corresponding to a TBD, only the periods* $q \equiv 1 \mod 2$ *are possible.*

For each tangle of a projection Π, the four points in the boundary are connected by Π following the three possible connection paths shown in Fig. 23.
If Π connects,
(1) NW to SW and NE to SE, we have the **V-connection path**,
(2) NW to NE and SW to SE, we have the **H-connection path**,
(3) NW to SE and NE to SW, we have the **X-connection path**.

Proof. The proof is straightforward by an examination of the possible connection paths (Fig. 23) on the tangles of the invariant TBD. In the case of knots, q cannot be even. □

3.4. Applications

(1) Seifert's algorithm applied to a q-periodic alternating projection of a knot K gives rise to a Seifert surface having the genus of K (see for instance [9]):

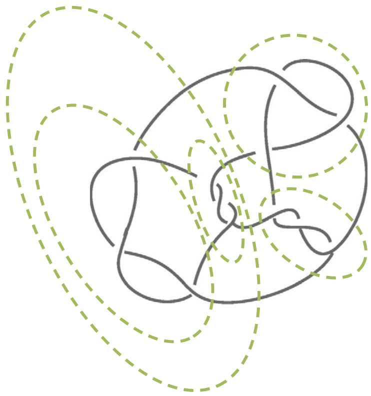

Fig. 24. Essential circles of $12a_{634}$.

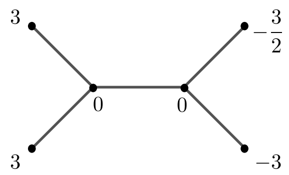

Fig. 25. Essential tree of $12a_{634}$.

Proposition 3.4. *(Particular case of [6]) There exists a q-equivariant orientable surface of K with minimal genus for* $q \geq 3$.

Proof. First consider an alternating projection Π of K which is invariant by a $2\pi/q$-rotation. Since Π is alternating then the Seifert surface produced using Seifert's algorithm on Π has minimal genus (this was first proven by Crowell and Murasugi, see a geometrical proof in [9]) and such a surface has an automorphism of order q which is produced by the $2\pi/q$-rotation on the projection sphere. □

(2) The crossing number of a knot is defined as the minimum number of crossings of all the possible projections of the knot. From Visibility Theorem 3.1, we have:

Proposition 3.5. *The crossing number of a prime alternating knot that is q-periodic with* $q \geq 3$ *is a multiple of* q.

Proof. Given a q-periodic alternating knot K with $q \geq 3$, by Visibility Theorem there is an alternating projection Π of K such that there is a $2\pi/q$-rotation leaving invariant Π. Since the orbit of any crossing in such a symmetric projection is of order q, the projection has its number of crossings a multiple of q. Note finally that the number of crossings of K is obtained on any reduced alternating projection (this fact was first shown by L. H. Kauffman, K. Murasugi and M. Thistlethwaite, see for instance [16]). □

(3) We now use the method in the proof of the Visibility Theorem 3.1 to study the 3-periodicity of the knot $12a_{634}$. We have:

Proposition 3.6. *The knot* $12a_{634}$ *is not q-periodic for* $q \geq 3$.

Proof. The essential decomposition of $12a_{634}$ is shown in Fig. 24 and the essential tree in Fig. 25. Since this tree has no symmetry different from the identity, the proposition follows from Corollary 3.2. □

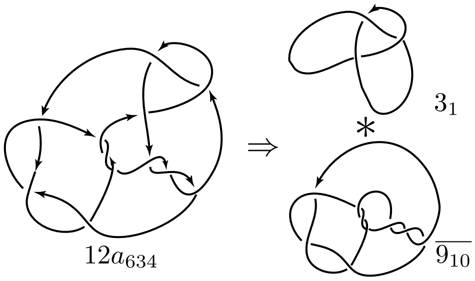

Fig. 26. $12a_{634} = 3_1 * \overline{9_{10}}$.

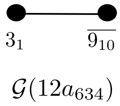

Fig. 27. Adjacency graph $\mathcal{G}(12a_{634})$.

With this result, like S. Jabuka and S. Naik [11], we thus complete the tabulation of the prime q-periodic alternating 12-crossings knots where q is an odd prime but our proof is not supported by computer calculations.

Conclusion. The knot $12a_{634}$ is chiral, non-invertible (see [16] and [17]) and not q-periodic for any $q \geq 3$.

(4) Murasugi decomposition of alternating knots and q-periodicity

As in the case of the essential structure tree, the adjacency graph presents symmetry when the corresponding knot is q-periodic. More precisely, we have:

Theorem 3.3. *Let K be a prime oriented alternating q-periodic knot with $q \geq 3$ and $A(K)$ its collection of Murasugi atoms. Then the adjacency graph of K admits an automorphism of order q and each atom of K is either q-periodic or it occurs a multiple of q times in $A(K)$.*

Proof. The Visibility Theorem 3.1 implies the existence of a q-periodic projection of K. Thus, the Theorem 3.2 follows from Theorem 1 in [5]. □

As an example, we apply Theorem 3.3 to the knot $12a_{634}$. With the notations and the definitions in [18], we have the Murasugi decomposition of $12a_{634}$ as:

$$12a_{634} = 3_1 * \overline{9_{10}}$$

where the knot $\overline{9_{10}}$ is the mirror image of 9_{10} (Fig. 26) and the adjacency graph $\mathcal{G}(12a_{634})$ is a tree with 2 vertices, one corresponding to the trefoil knot 3_1 and the other one to the knot $\overline{9_{10}}$ (Fig. 27).

Being a non-torus rational knot, the knot $\overline{9_{10}}$ is 2-periodic but not $q(\geq 3)$-periodic. Therefore Theorem 3.3 implies that $12a_{634}$ is not q-periodic with $q \geq 3$.

4. Addendum

There are overlapping results of the paper of K. Boyle [2] and this one. Both these papers use flypes and Flyping Theorem as the main tools, but differ in their techniques. The first version of this paper appeared in ArXiv earlier than [2].

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