BASIC ESTIMATES FOR SOLUTIONS OF A CLASS OF NONLOCAL ELLIPTIC AND PARABOLIC EQUATIONS

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Abstract. In this work we consider the problems

$$\begin{cases} \mathcal{L}u &= f & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and

$$\begin{cases} u_t + \mathcal{L}u &= f & \text{in } Q_T \equiv \Omega \times (0,T), \\ u(x,t) &= 0 & \text{in } \big(\mathbb{R}^N \setminus \Omega\big) \times (0,T), \\ u(x,0) &= 0 & \text{in } \Omega, \end{cases}$$

where $\mathcal{L}$ is a nonlocal differential operator and Ω is a bounded domain in $\mathbb{R}^N$, with Lipschitz boundary.

The main goal of this work is to study existence, uniqueness and summability of the solution u with respect to the summability of the datum f. In the process we establish an L^p-theory, for $p \geqslant 1$, associated to these problems and we prove some useful inequalities for the applications.

1. Introduction

The fractional Laplacian represents one of the simplest examples of pseudodifferential operators. From the original contributions by A. P. Calderón and A. Zygmund to the general theory developed by Nirenberg, Treves, Hörmander, Kohn, Fefferman and Beals among a large list of analysts, many researchers have contributed to the systematic study of the functional behavior of these operators. The theory was later extended to include symbols with less regular coefficients by Bony, Meyer, Sjöstrand and others. We refer to the books by L. Hörmander [30] and by M. Taylor [51] for complete presentations of this theory.

The concrete case of the fractional Laplacian has also appeared as a particular example of Levy-processes. Regarding this approach, there is a very extensive literature and we refer for instance to [6], [8], [32] and to the nice survey by E. Valdinoci [53].

In this work we are interested in some *integro-differential operators, with measurable and bounded coefficients,* that are non-local in nature. The study of problems in the framework of integro-differential equations is quite recent and has rised a great interest particularly in connection with problems involving nonlocal effects.

Non local operators naturally appear in elasticity problems [42], water waves [23], [24], [50], crystal dislocation [52], thin obstacle problems [20], phase transition

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[4], [18], [45], flames propagation [21], stratified materials [40], water waves [23], [24], [50], quasi-geostrophic flows [22], [34], among other phenomena.

In order to introduce our results, let us recall the definition of the fractional Laplacian. Denote by $\mathcal{S}(\mathbb{R}^N)$ the class of all Schwartz functions in $\mathbb{R}^N$. For $u, f \in \mathcal{S}(\mathbb{R}^N)$ the Fourier transform $\mathcal{F}$, properly normalized, provides the following equivalence

$$-\Delta u = f \qquad \Leftrightarrow \qquad |\xi|^2 \mathcal{F}(u)(\xi) = \mathcal{F}(f)(\xi).$$

With this observation at hand, it is natural to make the following definition.

For $0 < s < 1$, we define the fractional Laplacian $(-\Delta)^s$ as the operator given by the Fourier multiplier $|\xi|^{2s}$, that is, for $u \in \mathcal{S}(\mathbb{R}^N)$

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi).$$

A simple computation, that involves the inverse Fourier transform of a homogeneous tempered distribution, gives us the formal expression of the fractional Laplacian as an integral operator. More precisely, if $u \in \mathcal{S}(\mathbb{R}^N)$,

$$(-\Delta)^s u(x) := a_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \qquad s \in (0,1),$$

where

$$a_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1 - cos(\xi_1)}{|\xi|^{N+2s}} d\xi \right)^{-1} = 2^{2s-1} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|}. \tag{1.1}$$

The analytic theory of the fractional Laplacian in the whole space $\mathbb{R}^N$ is considered by now classical and can be seen, for instance, in Chapter 5 of [47]. Recently an extensive work, although far from exhaustive, has been developed for the associated Dirichlet problem in bounded domains. See, for instance, [8], [32], [39] and the references cited therein. Some results for semilinear equations can be seen, for instance, in [9], [10], [19], [41].

In this paper we consider a family of integro-differential operators related to the fractional Laplacian. The idea is to extend this operator to a class of operators with variable coefficients that give rise to nonlocal models parallel to the divergence-form local operator

$$-\text{div}\big(A(x)\nabla u\big). \tag{1.2}$$

Here $A(x)$ is a measurable, symmetric, bounded matrix valued function satisfying

$$\lambda |\xi|^2 \leqslant \langle A(x)\xi, \xi \rangle \leqslant \lambda^{-1} |\xi|^2,$$

for some $0 < \lambda < 1$.

We are interested in some integral expressions that correspond to the nonlocal version of the above. To reproduce the nonlocal effects, we consider the following class of kernels.

Definition 1.1. We say that the measurable function

$$\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(x,x) : x \in \mathbb{R}^N\} \to [0, \infty),$$

is a (symmetric) $2s$-*kernel* if it satisfies:

i) $\mathcal{K}(x,y) = \mathcal{K}(y,x)$.

ii) There exist $0 < s < 1$ and $0 < \lambda \leqslant 1$ such that for any $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, $x \neq y$,
$$\lambda \leqslant \mathcal{K}(x,y)|x-y|^{N+2s} \leqslant \lambda^{-1}.$$

Associated to every $2s-$kernel $\mathcal{K}(x, y)$ we define a principal value integral operator of the following form:

$$\mathcal{L}(u)(x) = \mathcal{L}_{\mathcal{K}}(u)(x) = P.V. \int_{\mathbb{R}^N} (u(x) - u(y))\mathcal{K}(x,y)\, dy\,. \tag{1.3}$$

Clearly, $\mathcal{L}$ is well defined if, say, $u \in \mathcal{S}(\mathbb{R}^N)$. Associated to $\mathcal{K}(x, y)$, and consequently to $\mathcal{L}$ too, we define the bilinear form

$$\mathcal{E} \,:\, \mathcal{S}(\mathbb{R}^N) \times \mathcal{S}(\mathbb{R}^N) \to \mathbb{R}\,,$$

as

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))(v(x)-v(y))\mathcal{K}(x,y)dx\,dy, \quad u, v \in \mathcal{S}(\mathbb{R}^N)\,. \tag{1.4}$$

Remark 1.2. It is worth pointing out that the class of kernels for which our results hold can be easily extended. For instance in [32] condition *ii)* is assumed for $|x - y| \leqslant 1$, while for $|x - y| > 1$ it suffices to have an estimate of the type

$$\mathcal{K}(x,y) \leqslant \frac{M}{|x-y|^{N+\alpha}},$$

for some $M \geqslant 1$ and some $\alpha > 0$. However, to keep it short we restrict ourselves just to the previous case.

1.1. **Organization of the article.** The paper has two parts. The first one is devoted to elliptic problems while the second deals with parabolic equations.

Given an operator $\mathcal{L}$ as above and given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, we study the Dirichlet problem,

$$\begin{cases} \mathcal{L}u &= f, \quad \text{in } \Omega, \\ \quad u &= 0, \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.5}$$

according with the summability of the datum f. We will assume that $N \geqslant 2$, although most of the results that we prove are also true for $N = 1$.

In Section 2, we briefly describe the natural functional framework for our problems. We begin recalling the definition of the fractional Sobolev space, $H^s_0(\Omega)$ and its embedding in Lebesgue spaces. We also formulate some interpolation theorems and we state some properties of the test functions that we will use.

Section 3 deals with the Dirichlet problem in the finite energy setting. Results related to (1.5) are disperse in the literature, specially when the datum f allows for a variational formulation of the problem, that is, when f is in the dual class of the associated fractional Sobolev space. In this section we collect some of them, old and new. We also obtain a fractional Picone's inequality with a nice application in mind, a Brezis-Oswald type Theorem (see [15] and [17]).

In Section 4 we work in a nonvariational setting, defining the concept of weak solution. The results for existence and regularity of weak solutions, that is, when f has a lower summability have been rarely treated for general kernels. Here we present several results related to the above questions in a systematic way. In this direction, the main results regarding problem (1.5) establish that the corresponding Calderón-Zygmund theory for the nonlocal operators holds similar to the one of the local case (1.2). This includes the estimates and summability properties of the solutions according with the summability of the data. In order to do that, we will develop in some cases new techniques.

The second part of the paper is devoted to the study of the associated parabolic problem,

$$\begin{cases} u_t + \mathcal{L}\, u &= f, & \text{in } Q_T, \\ u(x,t) &= 0, & \text{in } \big(\mathbb{R}^N \setminus \Omega\big) \times (0,T), \\ u(x,0) &= u_0(x) & \text{in } \Omega. \end{cases} \tag{1.6}$$

Here, the main goal is to establish the summability of u with respect to the summability of f. Some of our results in this context seem to be new even in the fractional Laplacian setting. In particular, we cover all the possible cases of regularity of f that in the local case have been studied, for example, in [13] and in [12] even in a nonlinear setting.

Section 5 is devoted to the parabolic case in the finite energy setting. We prove the existence and uniqueness result of the solution under variational hypotheses. In this context, Harnack's inequality (weak and strong) and Hölder's regularity can be seen in [27].

In Section 6 the existence and uniqueness of weak solution are established with integrable data, by using an approximation argument.

Section 7 considers the optimal summability of the solution in terms of the summability of the data. Our starting point here shows, in particular, an extension to the nonlocal case of the result obtained by Aronson-Serrin (see [7]).

Finally, in Section 8 we recover Kato's inequality in the elliptic and the parabolic problems.

2. The functional setting

First, we need to set a natural functional framework for our problems. According to the definition of the fractional Laplacian, it is natural to consider the following Hilbert space.

Definition 2.1. For $0 < s < 1$, we define the fractional Sobolev space of order s as

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N)\, s.t.\, |\xi|^s \mathcal{F}(u)(\xi) \in L^2(\mathbb{R}^N)\}.$$

Obviously, we can extend the operator $(-\Delta)^s u$ by density from $\mathcal{S}(\mathbb{R}^N)$ to $H^s(\mathbb{R}^N)$. Now, by Plancherel's identity, we obtain the expression of the norm as follows.

Proposition 2.2. *Let $N \geqslant 1$ and $0 < s < 1$. Then for all $u \in H^s(\mathbb{R}^N)$*

$$\int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}(u)(\xi)|^2 d\xi = a_{s,N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy, \tag{2.1}$$

where $a_{s,N}$ is the constant defined in (1.1)

See [28] for a detailed proof.

In this way, we obtain the associated scalar product as

$$\langle u, v \rangle_{H^s(\mathbb{R}^N)} = P.V. \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dx\, dy.$$

We can reformulate the previous result in the following useful way.

Proposition 2.3. *Let $s \in (0,1)$ and take $u \in H^s(\mathbb{R}^N)$. Then,*

$$\|u\|^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} u\,(-\Delta)^s u\,dx = 2a^{-1}_{N,s}\|(-\Delta)^{\frac{s}{2}}u\|^2_{L^2(\mathbb{R}^N)} = \||\xi|^s\mathcal{F}u\|^2_{L^2(\mathbb{R}^N)}.$$

As already observed, $a_{N,s}$ is a positive constant that vanishes for $s = 0, 1$. Since we are interested only in $s \in (0,1)$, we normalize the constant and we set it, without loss of generality, equal to one. Moreover, every time that we deal with a singular integral, we just write it omitting the 'P.V.' in front, for simplicity. Finally, in the estimates that follow, we deal with several constants: if they are not significant for the computations we call them c, taking into account that their value may vary from line to line.

Next, we recall several well known results about the spaces involved. The dual space of $H^s(\mathbb{R}^N)$ is

$$H^{-s}(\mathbb{R}^N) = \{f \in \mathcal{S}'(\mathbb{R}^N) \,/\, |\xi|^{-s}\mathcal{F}f \in L^2(\mathbb{R}^N)\}.$$

Let $\mathcal{L}$ be the operator associated with a $2s$-kernel like in (1.3). We have the following properties:

(1) $\mathcal{L} : H^s(\mathbb{R}^N) \to H^{-s}(\mathbb{R}^N)$ is a continuous operator.
(2) $\mathcal{L}$ is a symmetric operator in $H^s(\mathbb{R}^N)$, that is,
$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}v\rangle, \quad u, v \in H^s(\mathbb{R}^N).$$
(3) Denoting also by $\langle\cdot,\cdot\rangle$ the natural duality product between $H^s(\mathbb{R}^N)$ and $H^{-s}(\mathbb{R}^N)$, then
$$|\langle \mathcal{L}u, v\rangle| = |\mathcal{E}(u,v)| \leqslant \lambda^{-1}\|u\|_{H^s(\mathbb{R}^N)}\|v\|_{H^s(\mathbb{R}^N)}.$$
Consequently
$$\lambda\|u\|^2_{H^s(\mathbb{R}^N)} \leqslant \langle \mathcal{L}u, u\rangle \leqslant \lambda^{-1}\|u\|^2_{H^s(\mathbb{R}^N)}.$$

As already mentioned in the introduction, we are interested in considering Dirichlet problems in bounded Lipschitz domains $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$. Hence we need to define the space $H^s_0(\Omega)$ as

$$H^s_0(\Omega) = \{u \in H^s(\mathbb{R}^N) \text{ with } u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

endowed with the norm

$$\|u\|_{H^s_0(\Omega)} = \left(\iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\right)^{1/2},$$

where

$$D_\Omega = \mathbb{R}^N \times \mathbb{R}^N \setminus \left(\mathcal{C}\Omega \times \mathcal{C}\Omega\right).$$

The pair $(H^s_0(\Omega), ||\cdot||_{H^s_0(\Omega)})$ yields a Hilbert space (see for instance [41, Lemma 7] for more details). Notice that this norm is equivalent to the induced by $\mathcal{E}$ on $H^s_0(\Omega)$. The dual space of $H^s_0(\Omega)$ is

$$H^{-s}(\Omega) = \{f \in \mathcal{S}'(\mathbb{R}^N) \,/\, |\xi|^{-s}\mathcal{F}f \in L^2(\mathbb{R}^N)\}.$$

As above, we have that

$$\mathcal{L} : H^s_0(\Omega) \to H^{-s}(\Omega),$$

is a continuous operator.

Finally, we recall the Sobolev embedding theorem. We consider $\mathcal{D}^s(\mathbb{R}^N)$ the completion of $\mathcal{C}^\infty_0(\mathbb{R}^N)$ with respect to the norm $||(-\Delta)^{s/2}(\cdot)||_{L^2(\mathbb{R}^N)}$.

Theorem 2.4. *Let $s \in (0,1)$ and $N > 2s$. There exists a constant $\mathcal{S} = \mathcal{S}(N,s)$ such that, for any function $u \in \mathcal{D}^s(\mathbb{R}^N)$ we have*

$$\|u\|^2_{L^{2^*_s}(\mathbb{R}^N)} \leqslant \mathcal{S} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy,$$

where

$$\frac{1}{2^*_s} = \frac{1}{2} - \frac{s}{N}\,, \tag{2.2}$$

is the so called fractional critical Sobolev exponent. In particular, if $u \in H^s_0(\Omega)$ we have

$$\|u\|^2_{L^{2^*_s}(\Omega)} \leqslant \mathcal{S} \iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy, \tag{2.3}$$

with the same optimal constant $\mathcal{S}$.

Next, we recall for any $k \geqslant 0$ the definition of the functions $T_k(\sigma)$ and $G_k(\sigma)$, $\sigma \in \mathbb{R}^+$, that we often use in the paper:

$$T_k(\sigma) = \max\{-k; \min\{k,\sigma\}\} \quad \text{and} \quad G_k(\sigma) = \sigma - T_k(\sigma)\,. \tag{2.4}$$

For the reader convenience, the next results summarize the calculus properties of a linear nonlocal operator that will be used later.

The first result we want to state deals with a pointwise estimate that involves the action (by composition) of T_k and G_k on a measurable function.

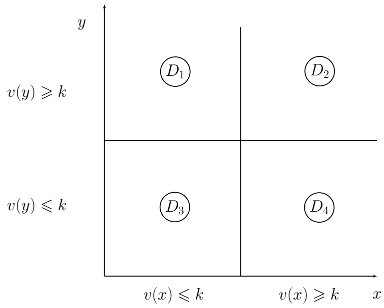
Lemma 2.5. *Let $v(x)$ be a positive measurable function in $\mathbb{R}^N$ and take any $\Omega \subseteq \mathbb{R}^N$. Then we have that*

$$\big(T_k(v(x)) - T_k(v(y))\big)\big(G_k(v(x)) - G_k(v(y))\big) \geqslant 0 \quad \text{a.e. in } D_\Omega\,.$$

Proof. First, let us decompose the set D_Ω in the following way:

$$D_\Omega = D_1 \cup D_2 \cup D_3 \cup D_4$$

where



$$D_1 = \{(x,y) \in D_\Omega \ : \ v(x) \leqslant k\,,\ v(y) \geqslant k\},$$
$$D_2 = \{(x,y) \in D_\Omega \ : \ v(x) \geqslant k\,,\ v(y) \geqslant k\},$$
$$D_3 = \{(x,y) \in D_\Omega \ : \ v(x) \leqslant k\,,\ v(y) \leqslant k\}$$

and

$$D_4 = \{(x,y) \in D_\Omega \ : \ v(x) \geqslant k\,,\ v(y) \leqslant k\}.$$

In fact, we show that

$$\big(T_k(v(x)) - T_k(v(y))\big)\big(G_k(v(x)) - G_k(v(y))\big) \begin{cases} = 0 & \text{in } D_2 \text{ and in } D_3, \\ \geqslant 0 & \text{in } D_1 \text{ and in } D_4. \end{cases}$$

By the pointwise definition of $T_k(s)$ and $G_k(s)$ we observe that

$$\big(T_k(v(x)) - T_k(v(y))\big)\big(G_k(v(x)) - G_k(v(y))\big) \equiv 0 \quad \text{in } D_2 \text{ and in } D_3\,.$$

On the other hand in D_1 we have that

$$\begin{aligned} &\big(T_k(v(x)) - T_k(v(y))\big)\big(G_k(v(x)) - G_k(v(y))\big)\chi_{D_1} \\ &= \big(v(x)-k\big)\big(k-v(y)\big)\chi_{D_1} = \big(k-v(x)\big)\ \big(v(y)-k\big) \geqslant 0\,. \end{aligned}$$

The inequality in D_4 follows in the same way. □

The above Lemma is very useful in order to get a priori estimates for solutions of Dirichlet problems associated to $\mathcal{L}$. In particular the following inequalities hold true.

Proposition 2.6. *Let v be a function in $H_0^s(\Omega)$:*

i) *if $\psi \in$ Lip $(\mathbb{R})$ is such that $\psi(0) = 0$, then $\psi(v) \in H_0^s(\Omega)$. In particular, for any $k \geqslant 0$, $T_k(v)$, $G_k(v) \in H_0^s(\Omega)$;*

ii) *for any $k \geqslant 0$*

$$\lambda \, \|G_k(v)\|^2_{H_0^s(\Omega)} \leqslant \int_\Omega G_k(v) \; \mathcal{L}v \, dx \, ; \tag{2.5}$$

iii) *for any $k \geqslant 0$*

$$\lambda \, \|T_k(v)\|^2_{H_0^s(\Omega)} \leqslant \int_\Omega T_k(v) \; \mathcal{L}v \, dx. \tag{2.6}$$

Proof.

i) Using the definition of Lipschitz function (L_ψ is the Lipschitz constant), it follows that

$$\begin{aligned}\|\psi(v)\|^2_{H_0^s(\Omega)} &= \iint_{D_\Omega} \frac{\big(\psi(v(x)) - \psi(v(y))\big)\big(\psi(v(x)) - \psi(v(y))\big)}{|x-y|^{N+2s}} dxdy \\ &\leqslant L_\psi^2 \int_{D_\Omega} \frac{\big(v(x) - v(x)\big)\big(v(x) - v(y)\big)}{|x-y|^{N+2s}} dxdy = L_\psi^2 \|v\|^2_{H_0^s(\Omega)} \, .\end{aligned}$$

ii) Due to i), for any $k \geqslant 0$, we are allowed to write the integral:

$$\int_\Omega G_k(v) \mathcal{L}v \, dx \; ;$$

hence we have that

$$\int_\Omega G_k(v) \mathcal{L}v \, dx = \iint_{D_\Omega} \big(v(x) - v(x)\big)\big(G_k(v(x)) - G_k(v(y))\big)\mathcal{K}(x,y) dxdy \, .$$

Notice that, since for any $\sigma \in \mathbb{R}$, $\sigma = T_k(\sigma) + G_k(\sigma)$, it follows

$$\begin{aligned}&\big(v(x) - v(x)\big)\big(G_k(v(x)) - G_k(v(y))\big) \\ &= \big(G_k(v(x)) - G_k(v(y))\big)^2 + \big(T_k(v(x)) - T_k(v(y))\big)\big(G_k(v(x)) - G_k(v(y))\big).\end{aligned}$$

By Lemma 2.5 we deduce that

$$\big(T_k(v(x)) - T_k(v(y))\big)\big(G_k(v(x)) - G_k(v(y))\big) \geqslant 0 \, ,$$

and consequently by (2.3), we have that

$$\begin{aligned}\lambda \, \|G_k(v)\|^2_{H_0^s(\Omega)} &\leqslant \iint_{D_\Omega} \big|G_k(v(x)) - G_k(v(y))\big|^2 \mathcal{K}(x,y) dxdy \\ &\leqslant \iint_{D_\Omega} \big(v(x) - v(x)\big)\big(G_k(v(x)) - G_k(v(y))\big)\mathcal{K}(x,y) dxdy = \int_\Omega G_k(v)\mathcal{L}v \, dx.\end{aligned}$$

iii) The idea of the proof is basically the same of ii). Let us consider for $k \geqslant 0$,

$$\int_\Omega T_k(v) \mathcal{L}v \, dx = \iint_{D_\Omega} \big(v(x) - v(x)\big)\big(T_k(v(x)) - T_k(v(y))\big)\mathcal{K}(x,y) dxdy \, .$$

Arguing as above we find that

$$\lambda \|T_k(v)\|^2_{H^s_0(\Omega)} \leqslant \iint_{D_\Omega} \big(T_k(v(x)) - T_k(v(y))\big)^2 k(x,y)dxdy$$
$$\leqslant \int_{D_\Omega} \big(v(x) - v(x)\big)\big(T_k(v(x)) - T_k(v(y))\big)\mathcal{K}(x,y)dxdy = \int_\Omega T_k(v)\mathcal{L}v\, dx\,.$$

□

We conclude this section by showing a computation that will be useful in what follows. We recall that when we deal with fractional derivatives, the chain rule does not hold true. As a substitute, if the function involved is convex or concave, an inequality holds. When applied to a fractional operator, we have the following elementary result.

Proposition 2.7. *Assume that* $\Phi : \mathbb{R} \to \mathbb{R}$ *is a Lipschitz convex function such that* $\Phi(0) = 0$*, then if* $u \in H^s_0(\Omega)$ *we have*

$$\mathcal{L}\Phi(u) \leqslant \Phi'(u)\,\mathcal{L}(u) \qquad a.e. \text{ in } \Omega. \tag{2.7}$$

On the other hand, if $\Psi : \mathbb{R} \to \mathbb{R}$ *is a Lipschitz concave function, then if* $u \in H^s_0(\Omega)$ *we have*

$$\mathcal{L}\Psi(u) \geqslant \Psi'(u)\,\mathcal{L}(u) \qquad a.e. \text{ in } \Omega. \tag{2.8}$$

Proof. The proof relies on elementary properties of convex and concave functions and the fact that the kernel defining $\mathcal{L}$ is non negative: to prove (2.7) we use that Φ satisfies $\Phi(a) - \Phi(b) \leqslant \Phi'(a)(a-b), \forall a, b$, whereas (2.8) follows from the opposite inequality for the concave function Ψ. □

A consequence of the previous result is the following Corollary that will be used in the next sections.

Corollary 2.8. *Consider a nonnegative function* $u \in H^s_0(\Omega)$*; then, for any positive* k*, it follows that*

$$\mathcal{L}T_k(u) \geqslant \mathcal{L}u \cdot \chi_{\{x\in\Omega,\, u(x)\leqslant k\}}, \quad x \in \Omega.$$

Proof. It suffices to apply (2.8) with the choice $\Phi(s) = T_k(s)$ that, restricted to positive values, turns out to be a concave function. □.

2.1. **Some interpolation results.** We will need the following classical *Interpolation Theorem* as a tool in the next sections.

In the statement the Marcinkiewicz spaces, $\mathcal{M}^p(\Omega)$, are involved. We recall that $\mathcal{M}^p(\Omega)$ coincides with the Lorenz space $L^{p,\infty}(\Omega)$, or "weak-$L^p(\Omega)$" (see for instance [49] for the definition).

Theorem 2.9. *Let* $1 < p_0, p_1, q_0, q_1 < \infty$ *and* $\theta \in (0,1)$*. Define* p *and* q *in the following way:*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad and \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If T *is a linear map with*

$$T : L^{p_0}(\Omega) \to L^{q_0}(\Omega) \qquad and \qquad T : \mathcal{M}^{p_1}(\Omega) \to L^{q_1}(\Omega)\,,$$

then we have

$$\|Tf\|_{L^p} \leqslant c\|f\|_{L^q}\,.$$

Proof. See for instance [48, 49]. □

Next, we recall the Gagliardo Nirenberg inequality, that plays a crucial role in the regularity estimates that we will prove in the parabolic framework.

Theorem 2.10 (Gagliardo-Nirenberg inequality)**.** *Let v be a function in $L^\rho(\mathbb{R}^N)$, such that $(-\Delta)^{\frac{s}{2}}v \in L^h(\mathbb{R}^N)$ with $h \geqslant 1$ and $\rho \geqslant 1$. Then there exists a positive constant c, depending only on N, s, h and ρ, such that*

$$\|v\|_{L^\eta(\mathbb{R}^N)} \leqslant c\|(-\Delta)^{\frac{s}{2}}v\|^{\theta}_{L^h(\mathbb{R}^N)}\|v\|^{1-\theta}_{L^\rho(\mathbb{R}^N)}, \tag{2.9}$$

for every η and θ satisfying

$$0 \leqslant \theta \leqslant 1, \qquad 1 \leqslant \eta < +\infty, \qquad \frac{1}{\eta} = \theta\left(\frac{1}{h} - \frac{s}{N}\right) + \frac{1-\theta}{\rho}.$$

For a complete account of the deep results about extensions of the Gagliardo-Nirenberg type inequalities to the fractional setting, one can see the paper by Brezis and Mironescu, [16]. In particular we are interested in the following result.

Theorem 2.11. *Assume that v belongs to $W^{s_1,p_1}(\mathbb{R}^N) \cap W^{s_2,p_2}(\mathbb{R}^N)$, for some s_i and $p_i > 0$, $i = 1, 2$. Then we have for $0 \leqslant s_1 < s_2 < \infty$, $1 < p_1 < \infty$, $1 < p_2 < \infty$,*

$$s = \theta s_1 + (1-\theta)s_2\,, \qquad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$$

that $\|v\|_{W^{s,p}(\mathbb{R}^N)} \leqslant c\|v\|_{W^{s_1,p_1}(\mathbb{R}^N)}\|v\|_{W^{s_2,p_2}(\mathbb{R}^N)}$.

Proof. See [16], Corollary 3.2. □

We collect here several results needed for the parabolic case. First we recall an interpolation result for parabolic Sobolev spaces.

Theorem 2.12. *Let $r_i, q_i \in (1,\infty]$ $i = 1, 2$ and suppose that $v \in L^{r_1}(0,T; L^{q_1}(\Omega)) \cap L^{r_2}(0,T; L^{q_2}(\Omega))$. Then for any $\theta \in (0,1)$, $u \in L^r(0,T; L^q(\Omega))$ with*

$$\frac{1}{r} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}. \tag{2.10}$$

and

$$\|v\|_{L^r(0,T;L^q(\Omega))} \leqslant \|v\|^{\theta}_{L^{r_1}(0,T;L^{q_1}(\Omega))}\|v\|^{1-\theta}_{L^{r_2}(0,T;L^{q_2}(\Omega))}\,.$$

Proof. See [7], Lemma 1. □

A first and immediate consequence of the previous result is the following embedding

$$\int_0^T\int_\Omega |v|^\sigma\,dx\,dt \leqslant c\|v\|_{L^\infty(0,T;L^\rho(\Omega))}\int_0^T\int_\Omega |(-\Delta)^{\frac{s}{2}}v|^h,$$

which holds for every function $v \in L^\infty(0,T; L^\rho(\Omega))$ such that $(-\Delta)^{\frac{s}{2}}v \in L^h(Q_T)$, with $h \geqslant 1$, $\rho \geqslant 1$ and $\sigma = \dfrac{h(N+\rho)}{N}$.

In particular, for $h = 2$, we get the following result.

Corollary 2.13. *Let v be a function in $L^2(0,T; H^s_0(\Omega)) \cap L^\infty(0,T; L^\rho(\Omega))$, with $\rho \geqslant 1$. Then v belongs to $L^m(Q)$, with $m = 2\frac{N+\rho}{N}$.*

A second consequence of (2.9) is the following embedding property.

Lemma 2.14. *Every function v belonging to $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^s(\Omega))$ belongs also to $L^\delta(0,T;L^\eta(\Omega))$ for every $\eta\in(2,2_s^*]$ and $\delta=\frac{4\eta s}{N(\eta-2)}$.*

Proof. Consider inequality (2.9) with $h=\rho=2$. Thus for any $\theta\in(0,1)$, one has that $\frac{1}{\eta}=\frac{1}{2}-\frac{\theta s}{N}$ for every fixed $\theta\in[0,1]$. Rising each term of inequality (2.9) to the power $\delta=\frac{2}{\theta}=\frac{4\eta s}{N(\eta-2)}$ and integrating in the time variable we deduce

$$\left(\int_0^T \|v\|^\delta_{L^\eta(\Omega)}dt\right)^{\frac{1}{\delta}} \leqslant c\left[\int_0^T \|(-\Delta)^{\frac{s}{2}}v\|^{\theta\delta}_{L^2(\mathbb{R}^N)}\|v\|^{(1-\theta)\delta}_{L^2(\Omega)}\right]^{\frac{1}{\delta}} \leqslant$$

$$c\|v\|^{1-\theta}_{L^\infty(0,T;L^2(\Omega))}\left[\int_0^T \|(-\Delta)^{\frac{s}{2}}v\|^{\theta\delta}_{L^2(\mathbb{R}^N)}dt\right]^{\frac{1}{\delta}},$$

from which the assertion follows . □

To conclude this section, we recall the following classical trace result.

Theorem 2.15. *Let H be a Hilbert space such that:*

$$V\overset{dense}{\hookrightarrow} H\hookrightarrow V'.$$

Let $u\in L^p(a,b;V)$ be such that u_t, defined in the distributional sense, belongs to $L^{p'}(a,b;V')$. Then u belongs to $C([a,b];H)$.

Proof. See [25], Chapter XVIII, Section 2, Theorem 1. □

3. Elliptic problem: Finite energy setting

Let us consider the following Dirichlet problem,

$$\begin{cases} \mathcal{L}u &= f, & \text{in } \Omega,\\ u &= 0, & \text{in } \mathbb{R}^N\setminus\Omega\,,\end{cases} \tag{3.1}$$

where $\Omega\subset\mathbb{R}^N$, $N\geqslant 2$ is a bounded domain with Lipschitz boundary, and $f\in H^{-s}(\Omega)$.

Here we recall the definition of a solution for such a problem.

Definition 3.1. For $f\in H^{-s}(\Omega)$ we say that $u\in H_0^s(\Omega)$ is a *finite energy solution* to (3.1) if

$$\mathcal{E}(u,w)=\langle f,w\rangle,\qquad \forall w\in H_0^s(\Omega)\,.$$

Remark 3.2. Observe that, since $L^{(2_s^*)'}(\Omega)\subset H^{-s}(\Omega)$, then $u\in H_0^s(\Omega)$ is a finite energy solution to (3.1) with $f\in L^{(2_s^*)'}(\Omega)$ if it satisfies that

$$\iint_{D_\Omega}\big(u(x)-u(x)\big)\big(w(x)-w(y)\big)\mathcal{K}(x,y)dxdy=\int_\Omega f(x)w(x)dx,\quad \forall w\in H_0^s(\Omega).$$

Due to the functional framework in which we set such a Dirichlet problem, we can state the first existence result for finite energy solutions.

Theorem 3.3. *Let $f\in H^{-s}(\Omega)$. Then there exists a unique $u\in H_0^s(\Omega)$ such that for all $v\in H_0^s(\Omega)$ the following identity holds,*

$$\mathcal{E}(u,v)=\langle f,v\rangle.$$

Moreover, if f belongs to $H^{-s}(\Omega)\cap L^1(\Omega)$ and it is positive, then u is positive, too.

Remark 3.4. In all the paper we assume that the datum f is positive and we deal with positive solutions. In order to see this fact in the energy framework, it suffices to choose $v = u^-$ as a test function (this is allowed thanks to Proposition 2.6) to deduce that $u^- \equiv 0$ in Ω.

In fact, such an assumption is not restrictive, since the general case can be obtained by decomposing the datum into its positive and negative part and then dealing with the two data separately, thanks to the linearity of the operator.

Proof. The existence and uniqueness part are consequences of the Lax-Milgram Theorem. □

Notice that the existence of a solution can also be obtained exploiting the variational structure of the equation. Indeed the equation in (3.1) turns out to be the Euler-Lagrange equation associated to the following (smooth enough) functional in $H_0^s(\Omega)$

$$J(u) = \frac{1}{2}\mathcal{E}(u,u) - \langle u, f\rangle,$$

where $\langle \cdot, \cdot \rangle$ represents the duality between $H_0^s(\Omega)$ and $H^{-s}(\Omega)$. Actually it is not hard to see that the above functional is lower bounded, coercive and weakly lower semicontinuous (with respect to the weak topology of $H_0^s(\Omega)$). Hence we can deduce the existence of a solution of (3.1) as a minimum of such a functional.

Thanks to the above result, we can define the following continuous map:

$$\begin{array}{rcl} T : H^{-s}(\Omega) & \to & H_0^s(\Omega) \\ f & \to & T(f) = u, \text{ the solution to (3.1)}. \end{array} \tag{3.2}$$

Notice that in the case of $\mathcal{L} = (-\Delta)^s$, then T happens to be the Green operator.

Once the existence and uniqueness in the energy framework is guaranteed, we turn into the study of the summability of the solution to the Dirichlet problem depending on the summability of the datum f.

3.1. **Bounded solutions: Moser and Stampacchia methods.**

In the local case in the seminal paper by G. Stampacchia [46] proves the boundedness of solutions by getting estimates on the level sets and by using a different type of arguments, non-linear test functions, J. Moser in [35] obtains the same type of results about the L^∞ estimates of the solutions.

In the fractional and probabilistic framework, the Stampacchia method has been used by Fukushima in [29]. Nevertheless, for the reader convenience, we will do the details for the result under a PDE formulation and we also give the method by Moser in the fractional setting.

The result that we prove is the following.

Theorem 3.5. *Let f belong to $L^m(\Omega)$, with $m > \frac{N}{2s}$. There exists a constant C, only depending on λ, N, Ω, $\|u\|_{H_0^s(\Omega)}$, $\|f\|_{L^m(\Omega)}$ and s, such that the unique energy solution of (3.1) satisfies*

$$\|u\|_{L^\infty(\Omega)} \leqslant C\,. \tag{3.3}$$

We present two proofs: the Moser proof and the Stampacchia proof.

3.1.1. **Boundedness of the solution via Moser method.**

Proof. Assume that $|\Omega| = 1$. The proof uses standard techniques for the fractional Laplacian. For $\beta > 1$ and $T > 0$ large, set the following convex function

$$\Phi(\sigma) = \Phi_T(\sigma) = \begin{cases} \sigma^\beta, & \text{if } \quad 0 \leqslant \sigma < T, \\ \beta T^{\beta-1}\sigma - (\beta-1)T^\beta, & \text{if } \quad \sigma \geqslant T. \end{cases}$$

Since Φ is Lipschitz (with constant $L_\Phi = \beta T^{\beta-1}$) and $\Phi(0) = 0$, then $\Phi(u) \in H_0^s(\Omega)$ and, using (2.7),

$$\mathcal{L}\Phi(u) \leqslant \Phi'(u)\mathcal{L}u. \tag{3.4}$$

Since Φ is positive, multiplying both sides of (3.4) by $\Phi(u)$ itself, we deduce

$$\lambda\mathcal{S}\|\Phi(u)\|^2_{L^{\frac{2N}{N-2s}}(\Omega)} \leqslant \lambda\ \|\Phi(u)\|^2_{H_0^s(\Omega)} \leqslant \int_\Omega \Phi(u)\mathcal{L}\Phi(u)\,.$$

On the other hand, using one more time the convexity of Φ,

$$\begin{aligned}\int_\Omega \Phi(u)\mathcal{L}\Phi(u) \leqslant \int_\Omega \Phi(u)\Phi'(u)\,\mathcal{L}u = \int_\Omega \Phi(u)\Phi'(u)\,f \\ \leqslant \quad \|f\|_{L^m(\Omega)}\ \|\Phi(u)\Phi'(u)\|_{L^{\frac{m}{m-1}}(\Omega)} \leqslant \quad \|f\|_{L^m(\Omega)}\ \|\beta\, u^{2\beta-1}\|_{L^{m'}(\Omega)},\end{aligned}$$

where we have used that $\Phi(r) \leqslant r^\beta$ and that $\Phi'(r) \leqslant \beta r^{\beta-1}$.

Combining both estimates, writing $p = \frac{2N}{N-2s}$ and $q = 2m'$, and letting $T \to \infty$ we arrive at

$$\begin{aligned}\|u\|_{L^{\beta p}(\Omega)} = \left(\int_\Omega u^{\beta p}\,dx\right)^{\frac{1}{\beta p}} &\leqslant \big(c\beta\|f\|_{L^m(\Omega)}\big)^{\frac{1}{2\beta}}\big(\textstyle\int_\Omega u^{\frac{2\beta-1}{2}q}\,dx\big)^{\frac{1}{\beta q}} \\ &\leqslant \big(c\beta\|f\|_{L^m(\Omega)}\big)^{\frac{1}{2\beta}}\big(1+\textstyle\int_\Omega u^{\beta q}\,dx\big)^{\frac{1}{\beta q}}.\end{aligned}$$

The rest is standard. We give the details for completeness.

Taking $\beta_0 = 1$, we define $\beta_j = \left(\frac{p}{q}\right)^j$ so that the following identity holds: $\beta_{j+1}q = \beta_j p$. Let us denote

$$A_j = \left(\int_\Omega u^{\beta_j p}\right)^{\frac{1}{\beta_j p}}, \qquad c_j = \big(c\|f\|_{L^m(\Omega)}\beta_j\big)^{\frac{1}{2\beta_j}}.$$

We have the recurrence formula $A_{j+1} \leqslant c_{j+1}\big(1 + A_j^{\beta_j p}\big)^{\frac{1}{\beta_j p}}$. By renormalizing if needed, we may assume that $A_0 = 1$ and then, since we are in a probability space, $A_j \geqslant 1, \forall j$. Taking logarithms,

$$\log A_{j+1} \leqslant \log c_{j+1} + \frac{1}{\beta_j p}\log\left(1 + A_j^{\beta_j p}\right) \leqslant \log c_{j+1} + \frac{1}{\beta_j p} + \log A_j,$$

where we have used that $\log(1+x) \leqslant 1 + \log x$, if $x \geqslant 1$. By induction,

$$\log A_{j+1} \leqslant \sum_{k=1}^{j+1}\log c_k + \sum_{k=0}^{j}\frac{1}{\beta_k p} + \log A_0.$$

Observe also that $A_0 \leqslant \|u\|_{H_0^s(\Omega)}$ and that the two series

$$\sum_{k=1}^{\infty}\log c_k = \sum_{k=1}^{\infty}\frac{1}{2\beta_k}\log(C\beta_k\ \|f\|_{L^m(\Omega)}), \qquad \sum_{k=1}^{\infty}\frac{1}{\beta_k p},$$

are convergent. This ends the proof since $\lim_{j\to\infty} A_j = \|u\|_{L^\infty(\Omega)}$. □

By Theorem 2.4, $L^{\frac{2N}{N+2s}}(\Omega) \subset H^{-s}(\Omega)$ and then if $f \in L^{\frac{2N}{N+2s}}(\Omega)$, u, the solution to the problem (3.1), satisfies that $u \in L^{\frac{2N}{N-2s}}(\Omega)$ and

$$\|u\|_{L^{\frac{2N}{N-2s}}(\Omega)} \leqslant c\|f\|_{L^{\frac{2N}{N+2s}}(\Omega)}.$$

3.1.2. **Boundedness of solution via Stampacchia's method.** In this subsection we give another proof of the boundedness of the solution of (3.1), for $f \in L^p(\Omega)$, $p > \frac{N}{2s}$. In this case the proof follows the idea of Stampacchia Theorem for second order elliptic equations with bounded coefficients.

The proof uses the following numerical iteration result whose proof is contained in [46].

Lemma 3.6. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function such that*

$$\psi(h) \leqslant \frac{M\,\psi(k)^\delta}{(h-k)^\gamma}, \quad \forall h > k > 0,$$

where $M > 0$, $\delta > 1$ and $\gamma > 0$. Then $\psi(d) = 0$, where $d^\gamma = M\,\psi(0)^{\delta-1}\,2^{\frac{\delta\gamma}{\delta-1}}$.

Proof [of Theorem 3.5] Let $k > 0$ and choose $w = G_k(u)$ as a test function in the energy formulation of (3.1). Hence, thanks to (2.5), we deduce that

$$\iint_{D_\Omega} \big|G_k(u(x)) - G_k(u(y))\big|^2 \mathcal{K}(x,y)dxdy \leqslant \int_{A_k} f\,G_k(u(x))dx$$

where $A_k = \{x \in \Omega \,:\, u \geqslant k\}$. Applying Sobolev inequality in the left hand side,

$$\lambda\mathcal{S}^{-2}\|G_k(u)\|^2_{L^{2^*_s}(\Omega)} \leqslant \iint_{D_\Omega} \big|G_k(u(x)) - G_k(u(y))\big|^2 \mathcal{K}(x,y)dxdy\,,$$

and Hölder inequality in the right hand side,

$$\left|\int_{A_k} f\,G_k(u(x))dx\right| \leqslant \|f\|_{L^m(\Omega)}\|G_k(u)\|_{L^{2^*_s}(\Omega)}|A_k|^{1-\frac{1}{2^*_s}-\frac{1}{m}}$$

we have that

$$\lambda\,\|G_k(u)\|_{L^{2^*_s}(\Omega)} \leqslant \mathcal{S}^2\|f\|_{L^m(\Omega)}|A_k|^{1-\frac{1}{2^*_s}-\frac{1}{m}}\,.$$

Consequently, for any $h > k$, we have that $A_h \subset A_k$ and $G_k(s)\chi_{A_h} \geqslant (h-k)$ and thus

$$(h-k)|A_h|^{\frac{1}{2^*_s}} \leqslant \frac{\mathcal{S}^2}{\lambda}\|f\|_{L^m(\Omega)}|A_k|^{1-\frac{1}{2^*_s}-\frac{1}{m}}.$$

Manipulating the above inequality we deduce that

$$|A_h| \leqslant \frac{\mathcal{S}^{2\ 2^*_s}\|f\|^{2^*_s}_{L^m(\Omega)}|A_k|^{2^*_s(1-\frac{1}{2^*_s}-\frac{1}{m})}}{\lambda^{2^*_s}(h-k)^{2^*_s}}\,.$$

Since $m > \frac{N}{2s}$ we have that

$$2^*_s\Big(1-\frac{1}{2^*_s}-\frac{1}{m}\Big) > 1.$$

Hence we apply Lemma 3.6 with the choice $\psi(\sigma) = |A_\sigma|$, consequently there exists k_0 such that $\psi(k) \equiv 0$ for any $k \geqslant k_0$ and thus $\operatorname{ess\,sup}_\Omega\ u \leqslant k_0$. □

Remark 3.7. Observe that, in fact, solutions are more than bounded. Indeed in [44] it is obtained a $\mathcal{C}^\beta(\Omega)$ regularity of the bounded solutions associated to the the Dirichlet problem (3.1), for $f \in L^m(\Omega)$, $m > N/2s$. In such a result, $\beta \in (0,1)$ only depends on the structural constants of $\mathcal{L}$ (see also [32] for the case of $\mathcal{L}$-harmonic functions).

3.2. **The limit case $m = \frac{N}{2s}$: exponential summability.** As in the local case, we stress that for the limit case $f \in L^{N/2s}(\Omega)$ boundedness of solutions cannot be expected, but solutions satisfy an exponential summability. Here we state the result for such a case.

Theorem 3.8. *Assume that $f \in L^{\frac{N}{2s}}(\Omega)$, then the unique energy solution u to* (3.1) *satisfies the following estimate:*

$$\exists\, \alpha > 0 \;\; s.t. \qquad \int_\Omega e^{\alpha u}\, dx < \infty. \tag{3.5}$$

In particular, $u \in L^q(\Omega)$ for any $q \geqslant 1$.

Proof. In order to prove the regularity of the energy solution (the existence follows by Theorem 3.3), let us consider for any $T > 0$, the following convex function:

$$\Phi(\sigma) = \Phi_T(\sigma) = \begin{cases} e^{\alpha\sigma} - 1, & \text{if} \quad 0 \leqslant \sigma < T \\ \alpha e^{\alpha T}(\sigma - T) + e^{\alpha T} - 1, & \text{if} \quad \sigma \geqslant T \end{cases}$$

where $\alpha > 0$ will be fixed later. Let u be the unique $H_0^s(\Omega)$ solution to (3.1) and since $\Phi_T(\sigma)$ is a convex and globally Lipschitz function with $\Phi_T(0) = 0$, then $\Phi_T(u)$ is an admissible test function. Thus, according with (2.3) and (2.7) we deduce that

$$\lambda \mathcal{S} \|\Phi(u)\|^2_{L^{2_s^*}(\Omega)} \leqslant \int_\Omega \Phi'(u)\Phi(u)\, \mathcal{L}u = \int_\Omega \Phi'(u)\Phi(u)\, f. \tag{3.6}$$

Observe that for $u < T$ we have $\Phi'(u) = \alpha e^{\alpha u} = \alpha\Phi(u) + \alpha$, and so by (3.6) we split the last integral in two parts:

$$\begin{aligned} \lambda \mathcal{S} \|\Phi(u)\|^2_{L^{2_s^*}(\Omega)} &\leqslant \alpha \int_{\{x\in\Omega\,:\,u<T\}} \big[\Phi(u)^2 f + \Phi(u)\, f\big] dx \\ &\quad + \alpha\, e^{\alpha T} \int_{\{x\in\Omega\,:\,u\geqslant T\}} \Phi(u)\, f dx\,. \end{aligned} \tag{3.7}$$

Using Hölder's inequality we have that

$$\alpha \int_{\{x\in\Omega\,:\,u<T\}} \big[\Phi(u)^2 f + \Phi(u)\, f\big] dx \leqslant \alpha \|\Phi(u)\|^2_{L^{2_s^*}(\Omega)} \|f\|_{L^{\frac{N}{2s}}(\Omega)} (1 + |\Omega|^{\frac{N-2s}{2N}}) + c(f)\,,$$

and moreover, since Φ is increasing, we get

$$\alpha e^{\alpha T} \int_{\{x\in\Omega\,:\,u\geqslant T\}} \Phi(u)\, f dx \leqslant \alpha \frac{e^{\alpha T}}{\Phi(T)} \int_{\{u\geqslant T\}} \Phi(u)^2 f \leqslant 2\alpha \|\Phi(u)\|^2_{L^{2_s^*}(\Omega)} \|f\|_{L^{\frac{N}{2s}}(\Omega)},$$

provided that $\alpha\, T > 1$. Putting together the above estimates in (3.7), and fixing α small enough, we get a bound for the $L^{2_s^*}(\Omega)$ norm of $\Phi(u)$, that implies (3.5), once one takes T sufficiently large (depending on f, but not on u). $\square$

3.3. **A Calderón-Zygmund type result.** In the finite energy setting we can obtain an optimal result on summability that extends to the fractional case the Calderón-Zygmund theory. More precisely we have the following result.

Theorem 3.9. *Let f be a positive function that belongs to $L^m(\Omega)$, with $\frac{2N}{N+2s} \leqslant m < \frac{N}{2s}$. Then, there exists a constant $c = c(N, m, s) > 0$ such that the unique energy solution to*

$$\mathcal{L}u = f \quad \text{in } \Omega, \qquad u \in H_0^s(\Omega)\,,$$

satisfies,

$$\|u\|_{L^{m_s^{**}}(\Omega)} \leqslant c\,\|f\|_{L^m(\Omega)} \qquad \text{where} \qquad m_s^{**} = \frac{mN}{N-2ms}\,. \tag{3.8}$$

Proof. The existence of a unique energy solution was obtained in Theorem 3.3. Hence we have just to deal with the regularity of the solution.

Define the following convex and differentiable function

$$\Phi(\sigma) = \begin{cases} \sigma^\beta & \text{if } 0 \leqslant \sigma \leqslant T \\ \beta T^{\beta-1}(\sigma - T) + T^\beta & \text{if } \sigma > T\,, \end{cases} \tag{3.9}$$

with $\beta = \frac{m_s^{**}}{2_s^*} > 1$.

Notice that since Φ is convex and $\Phi(0) = 0$, then, according to Proposition 2.6, $\Phi(u)$ is an admissible test function.

Hence by (2.7) we have that

$$\mathcal{L}\Phi(u) \leqslant f(x)\Phi'(u)$$

and multiplying the above inequality one more time by $\Phi(u)$, we deduce that

$$\mathcal{E}(\Phi(u), \Phi(u)) \leqslant \int_\Omega f\Phi(u)\Phi'(u)dx\,.$$

It is not hard to see that there exists a positive constant c independent of T, such that

$$\Phi(u)\Phi'(u) \leqslant c\Phi(u)^{2_s^*}\,.$$

Thus, using that $\frac{2}{2_s^*} > \frac{1}{m'}$, and by Sobolev inequality, we deduce that (3.8) holds true. □

3.4. **Further fractional regularity.** This section is devoted to prove an interesting property of the energy solutions of (3.1).

Indeed we observe that any function in $H_0^s(\Omega)$, thanks to the smoothness of the boundary of Ω, is such that its extension (as 0 outside Ω) belongs to $W^{s,2}(\mathbb{R}^N)$. Thus, we can apply the interpolation result that we stated in Theorem 2.11 and we get the following regularity result.

Theorem 3.10. *Let u be an energy solution of (3.1) with f in $L^m(\Omega)$, $m \geqslant (2_s^*)'$. Then u belongs to $W_0^{\sigma,p}(\Omega)$ where*

$$\sigma = \theta s \qquad \text{and} \qquad \frac{1}{p} = \frac{1}{m} + \theta\Big(\frac{1}{2} - \frac{1}{m}\Big) - \frac{2s}{N}(1-\theta) \quad \theta \in (0,1)\,. \tag{3.10}$$

Observe that in the above theorem the Sobolev space $W_0^{\sigma,p}(\Omega)$, with $\sigma \in (0,1)$ and $p > 1$ is involved; for the definition of such a space see [2].

Proof. It easily follows by applying Theorem 2.11, interpolating between the Sobolev space $H_0^s(\Omega) \equiv W_0^{s,2}(\Omega)$ (i.e. $\theta = 1$) and $L^{m_s^{**}}(\Omega)$ (i.e. $\theta = 0$). □

Observe that the regularity given by the above Theorem do not appear in the literature of the local case (s=1), where one look for estimates either on u or on ∇u.

3.5. **A fractional Picone's inequality and applications.** We formulate an extension of an inequality obtained by Picone in [37]. To be more precise, Picone considers that if $u,\, v \in \mathcal{C}^2(a,b)$ and $u > \delta > 0$ in (a,b) then,

$$(v')^2 \geqslant \big(\frac{v^2}{u}\big)' u'.$$

The corresponding extension of this pointwise inequality to higher dimensions is that if $\Omega \subset \mathbb{R}$ and $u,\, v \in \mathcal{C}^2(\Omega)$ and $u > \delta > 0$ in Ω then

$$|\nabla v|^2 \geqslant \langle \nabla\big(\frac{v^2}{u}\big), \nabla u\rangle$$

and for any $1 < p < \infty$,

$$|\nabla v|^p \geqslant \langle \nabla\big(\frac{v^p}{u^{p-1}}\big), |\nabla u|^{p-2}\nabla u\rangle,$$

see [1] and [5].

The useful integral form was obtained in a quite general framework in [1], where also some interesting applications were obtained. It is precisely this integral form the one that we prove for the fractional operator $\mathcal{L}$. More precisely, we obtain the following result.

Theorem 3.11. *(Picone's Inequality.) Consider $u, v \in H_0^s(\Omega)$, where $\mathcal{L}u$ is a positive bounded Radon measure in Ω, and $u \gneqq 0$. Then,*

$$\int_\Omega \frac{\mathcal{L}u}{u} v^2\, dx \leqslant \mathcal{E}(v,v). \tag{3.11}$$

Proof. Let us recall first that for any $\phi, \psi \in H_0^s(\Omega)$, we have, thanks to the symmetry of $\mathcal{L}$, that

$$\int_\Omega \phi\ \mathcal{L}\psi\ dx = \int_\Omega \mathcal{L}\phi\ \psi\ dx = \mathcal{E}(\phi,\psi) \leqslant \lambda^{-1}\|\phi\|_{H_0^s(\Omega)}\|\psi\|_{H_0^s(\Omega)}. \tag{3.12}$$

We set, for any $k, \eta > 0$, $v_k = T_k(v)$, $\tilde{u} = u + \eta$ and we define $w = \dfrac{v_k^2}{\tilde{u}}$; actually it is not hard to see that $w \in H_0^s(\Omega)$. We want to prove that, $\forall u, v \in H_0^s(\Omega)$ and $\forall k, \eta > 0$, and thanks to (3.12),

$$\mathcal{E}(u,w) = \int \mathcal{L}u \frac{v_k^2}{\tilde{u}} \leqslant \mathcal{E}(v_k, v_k) \leqslant \lambda^{-1}\|T_k(v)\|^2_{H_0^s}. \tag{3.13}$$

Once we have proved such an inequality, (3.11) follows by letting k diverge and η vanish, using the monotone convergence theorem and Fatou's Lemma (this is why we ask $\mathcal{L}u$ to be a positive Radon measure). Observe that

$$\begin{aligned}(u(x)-u(y))(w(x)-w(y)) &= \big(\tilde{u}(x)-\tilde{u}(y)\big)\left(\frac{v_k^2(x)}{\tilde{u}(x)} - \frac{v_k^2(y)}{\tilde{u}(y)}\right)\\ &= v_k^2(x) + v_k^2(y) - v_k^2(x)\frac{\tilde{u}(y)}{\tilde{u}(x)} - v_k^2(y)\frac{\tilde{u}(x)}{\tilde{u}(y)}.\end{aligned} \tag{3.14}$$

Thus, by Young inequality we deduce that

$$v_k^2(x)\frac{\tilde{u}(y)}{\tilde{u}(x)} + v_k^2(y)\frac{\tilde{u}(x)}{\tilde{u}(y)} \geqslant 2v_k(y)v_k(x)$$

and so, gathering together the above inequality with (3.14), we get

$$(u(x)-u(y))(w(x)-w(y)) \leqslant \big(v_k(x)-v_k(y)\big)^2 .$$

Hence (3.13) follows by changing the above inequality into the corresponding integral one weighed with the kernel $\mathcal{K}(x,y)$. □

Next we present an application of Picone's inequality to a semilinear problem first studied by Brezis-Oswald in [17], and then by Brezis-Kamin in [15]. To be more precise, consider the semilinear problem

$$\begin{cases} \mathcal{L}u = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u \equiv 0 & \text{in } \mathbb{R}^N \setminus \Omega\,, \end{cases} \tag{3.15}$$

where $f : \Omega \times [0,\infty) \to \mathbb{R}$ is a Caratheodory function. The aim is to give some condition on f to have uniqueness of solutions to (3.15).

Observe that if $\mathcal{L}$ is a linear differential operator of second order, this problem is classical and it has been solved by H. Brezis and L. Oswald in 1986, 3.15. Indeed, requiring that the nonlinearity satisfies that

$$s \to \frac{f(x,s)}{s} \quad \text{is decreasing for a.e. } x \in \Omega, \tag{3.16}$$

they prove uniqueness for solutions of

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Here we prove the analogous for the problem (3.15), associated to the operator $\mathcal{L}$. First, we need to recall the following definition.

Definition 3.12. A sub solution (super solution) to (3.15) is a positive function $u \in H_0^s(\Omega) \cap L^\infty(\Omega)$ such that $f(x,u) \in L^1(\Omega)$ and

$$\int_\Omega u\, \mathcal{L}\varphi \leqslant (\text{respectively } \geqslant) \int_\Omega f(x,u)\varphi\,, \qquad \forall \varphi \in H_0^s(\Omega)\,, \quad \varphi \geqslant 0\,. \tag{3.17}$$

A solution is both a sub solution and a super solution.

Now we an state our Brezis-Kamin-Oswald type result.

Theorem 3.13. *Let u and v be a sub and a super solution, respectively. Assume that (3.16) holds true, then $u \leqslant v$ a.e. in Ω.*

Remark 3.14. Before proving the uniqueness result, observe that solutions of problem (3.15) exist. Indeed the equation in (3.15) has a variational structure, i.e. solutions of such a problem are critical points of the functional

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} \mathcal{E}(v,v)dx - \int_\Omega F(x,v)dx, \qquad \forall v \in H_0^s(\Omega),$$

where $F(x,\sigma) = \int_0^\sigma f(x,\tau)d\tau$. Thanks to the hypothesis on $f(x,\sigma)$, then $F(x,\sigma)$ is smooth and has at most a sub quadratic behavior at infinity. Thus, the functional $I(v)$ is coercive and lower weakly semicontinuous (with respect to the topology of $H_0^s(\Omega)$) and consequently the minimum is an energy solution of (3.15). By standard bootstrap arguments it is also bounded and satisfies the formulation of the Definition 3.12.

Proof. Let us choose $\Big(u(x) - \frac{v^2(x)}{u(x)}\Big)^+$ as a test function in the formulation of u and $\Big(\frac{u^2(x)}{v(x)} - v(x)\Big)^+$ in the formulation of v (these choices are admissible due to Picone's inequality). Thus, subtracting the two formulations, we get:

$$\begin{aligned}&\int_\Omega u\mathcal{L}\Big(u - \frac{v^2}{u}\Big)^+ dx - \int_\Omega v\mathcal{L}\Big(\frac{u^2}{v} - v\Big)^+ dx\\ \leqslant &\int_\Omega f(x,u)\Big(u(x) - \frac{v^2(x)}{u(x)}\Big)^+ dx - \int_\Omega f(x,v)\Big(\frac{u^2(x)}{v(x)} - v(x)\Big)^+ dx.\end{aligned} \tag{3.18}$$

Notice that the right hand side is negative thanks to (3.16), so that we get rid of the left hand side. Observe that developing the computations of the left hand side, we get, denoting $\Omega^+ = \{x \in \Omega \,:\, u(x) \geqslant v(x)\}$,

$$\begin{aligned}&\int_{D_{\Omega^+}} \big(u(x)-u(y)\big)\Big(u(x) - \frac{v^2(x)}{u(x)} - u(y) + \frac{v^2(y)}{u(y)}\Big)\,\mathcal{K}(x,y)dxdy\\ &-\int_{D_{\Omega^+}} \big(v(x)-v(y)\big)\Big(\frac{u^2(x)}{v(x)} - v(x) - \frac{u^2(y)}{v(y)} + v(y)\Big)\,\mathcal{K}(x,y)dxdy\\ &\quad= -\int_{\Omega^+} \mathcal{L}u\,\frac{v^2}{u}\,dx + \mathcal{E}(u,u) - \int_{\Omega^+} \mathcal{L}v\,\frac{u^2}{v}\,dx + \mathcal{E}(v,v)\,,\end{aligned}$$

and the last term is positive thanks to Picone Inequality. Consequently $u \leqslant v$ a.e. in Ω. □

4. Nonvariational setting for elliptic problems: weak solutions

When $f \in L^m(\Omega)$ with $1 \leqslant m < \frac{2N}{N+2s}$ we cannot expect a solution of finite energy. However, as in the local case, by relaxing the meaning of solution we can show the existence of weaker solution for nonvariational data. Consider the problem

$$\left\{\begin{array}{rcll}\mathcal{L}u &=& f & \text{in } \Omega\\ u &=& 0 & \text{in } \mathbb{R}^N \setminus \Omega.\end{array}\right. \tag{4.1}$$

In the case of $\Omega = \mathbb{R}^N$ the existence and uniqueness of *renormalized solutions* is obtained in [3] (see also [31]). However we consider a *weak solution* to (4.1) in a very much elementary context, in bounded domains and making explicit the summability of the solution.

Definition 4.1. We define the class of test functions

$$\mathcal{T}(\Omega) = \{\phi \mid \mathcal{L}(\phi) = \psi \text{ in } \Omega, \quad \phi = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \quad \psi \in \mathcal{C}_0^\infty(\Omega)\}. \tag{4.2}$$

Notice that if $v \in \mathcal{T}(\Omega)$ then, using the results in the previous section, $v \in H_0^s(\Omega) \cap L^\infty(\Omega)$. Moreover, according to the regularity theory developed in [43], if Ω is smooth enough, there exists a constant $\beta > 0$ (that depends only on the structural constants) such that $v \in \mathcal{C}^\beta(\Omega)$ (see also [32]).

Definition 4.2. We say that $u \in L^1(\Omega)$ is a *weak solution* to (4.1) if for $f \in L^1(\Omega)$ we have that

$$\int_\Omega u\psi dx = \int_\Omega f\phi dx,$$

for any $\phi \in \mathcal{T}(\Omega)$ with $\psi \in \mathcal{C}_0^\infty(\Omega)$.

Remark 4.3. We recall that if in the definition of the test function space, we consider $\psi \in L^m(\Omega)$, $m > N/2$, we recover the classical definition of *duality solution* by Stampacchia (see [46]).
Indeed, using Theorem 3.5, ϕ is bounded and consequently all the terms in the above identity make sense.

We also stress that such a definition can be extended to equations with measure data, if we have a regularity result that guarantees the continuity up to the boundary of the solution of the problem $\mathcal{L}(\phi) = \psi$ in Ω, $\quad \phi = 0$ in $\mathbb{R}^N \setminus \Omega$. Results in this direction can be found in [31].

Now we can state the existence result for L^1-data.

Theorem 4.4. *For any f belonging to $L^1(\Omega)$ there exists a unique weak solution to (4.1). Moreover,*

$$\forall k \geqslant 0 \qquad T_k(u) \in H_0^s(\Omega)\,, \tag{4.3}$$

$$u \in L^q(\Omega)\,, \qquad \forall\, q \in \big(1, \frac{N}{N-2s}\big) \tag{4.4}$$

and

$$\big|(-\Delta)^{\frac{s}{2}} u\big| \in L^r(\Omega)\,, \qquad \forall\, r \in \big(1, \frac{N}{N-s}\big)\,. \tag{4.5}$$

Remark 4.5. The philosophy behind the above result follows the ideas that come from the analogous in the local case (see, for instance [46] and [12] for the nonlinear case). Indeed, as already observed, we cannot expect solutions to belong to the Sobolev space $H_0^s(\Omega)$ since $L^1(\Omega)$ is not embedded into $H^{-s}(\Omega)$. Anyway, we prove that any truncation at level k does belong to the energy space.
On the other hand, observe that if $s = 1$ we recover the existence and regularity results that are, nowadays, well known for second order elliptic operator with L^1 data. Indeed in such a case, according to the results mentioned above, weak solutions u are such that $|\nabla u| \in L^r(\Omega)$, $\forall r \in \big(1, \frac{N}{N-1}\big)$, and $u \in L^q(\Omega)$, $\forall q \in \big(1, \frac{N}{N-2}\big)$.

Remark 4.6. Observe that the two estimates (4.4) and (4.5) are essentially different one from the other. Indeed u has been defined as zero outside Ω so that (4.4) turns out to be a global estimate on u. On the other hand, the a priori estimate on the $s/2$-derivative of u cannot be extended to a global one. Indeed the nonlocal nature of the equation makes $\big|(-\Delta)^{\frac{s}{2}} u\big|$ to be different from zero in $\mathbb{R}^N \setminus \Omega$, but since the (pseudodifferential) equation is set only in Ω we cannot get summability information outside Ω.

Proof. The proof of the above theorem is split into two parts. We first prove uniqueness of a weak solution. We stress that the proof follows the idea of the one by Stampacchia. Subsequently we construct the solution by approximation with solutions of problems with smooth data. We prove suitable a priori estimates (that, in particular, imply (4.3)–(4.5)) and compactness of the approximating sequence in $L^1(\Omega)$.

Uniqueness.- Assume that u is a weak solution to (4.1) with $f = 0$, then

$$\int_\Omega u\psi\; dx = 0 \qquad \text{for any} \quad \psi \in \mathcal{C}_0^\infty(\Omega).$$

Therefore $u \equiv 0$.

Existence.- We obtain the solution to (4.1) as a limit of solutions to approximated problems.

Consider $f_n \in L^\infty(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$ and let u_n be the solution to the problem

$$\begin{cases} \mathcal{L}u_n = f_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.6}$$

Then we prove the existence of a solution in several steps.

Step 1. *There exists a positive constant c, only depending on N, Ω and s, such that*

$$\|u_n\|_{L^q(\Omega)} \leqslant c\,\|f_n\|_{L^1(\Omega)}, \qquad \forall q \in \big(1, \frac{N}{N-2s}\big)\,. \tag{4.7}$$

Let us multiply the equation in (4.6) by $T_k(u_n)$, for $k \geqslant 0$, and let us integrate over Ω. Thanks to Proposition 2.6 we obtain

$$\lambda\,\|T_k(u_n)\|^2_{L^{2^*_s}(\Omega)} \leqslant \mathcal{S}^2\,k\|f_n\|_{L^1(\Omega)}\,. \tag{4.8}$$

Moreover by using the Sobolev inequality and an elementary estimate we have that

$$\lambda\,k^2 \text{ meas }(A_{n,k}(u_n))^{\frac{N-2s}{N}} \leqslant \lambda\,\|T_k(u_n)\|^2_{L^{2^*_s}(\Omega)} \leqslant \mathcal{S}^2\,k\|f_n\|_{L^1(\Omega)}\,,$$

where $A_{n,k}(u_n) = \{x \in \Omega\,:\,u_n(x) \geqslant k\}$. It follows that

$$\text{meas }(A_{n,k}(u_n)) \leqslant c\Big(\frac{\|f_n\|_{L^1(\Omega)}}{k}\Big)^{\frac{N}{N-2s}}\,. \tag{4.9}$$

It means that u_n is bounded in the Marcinkiewicz space $\mathcal{M}^{\frac{N}{N-2s}}(\Omega)$ and consequently (4.7) holds true.

Step 2. *There exists a positive constant c, just depending on q, N, Ω and s, such that*

$$\|(-\Delta)^{\frac{s}{2}}u_n\|_{L^r(\Omega)} \leqslant c\,\|f_n\|_{L^1(\Omega)}, \qquad \forall r \in \big(1, \frac{N}{N-s}\big). \tag{4.10}$$

We fix $\lambda > 0$, and, for any positive k, we want to estimate the measure of the following set:

$$\begin{aligned} &\big\{x \in \Omega\,:\,|(-\Delta)^{\frac{s}{2}}u_n| \geqslant \lambda\big\} \\ =&\big\{x \in \Omega\,:\,|(-\Delta)^{\frac{s}{2}}u_n| \geqslant \lambda, u_n < k\big\} \cup \big\{x \in \Omega\,:\,|(-\Delta)^{\frac{s}{2}}u_n| \geqslant \lambda, u_n \geqslant k\big\}, \end{aligned}$$

and consequently

$$\big\{x \in \Omega\,:\,|(-\Delta)^{\frac{s}{2}}u_n| \geqslant \lambda\big\} \subset \big\{x \in \Omega\,:\,|(-\Delta)^{\frac{s}{2}}u_n| \geqslant \lambda, u_n < k\big\} \cup A_{n,k}(u_n).$$

Since

$$\text{meas }(\big\{x \in \Omega\,:\,|(-\Delta)^{\frac{s}{2}}u_n| \geqslant \lambda,\ u_n < k\big\}) \leqslant \frac{1}{\lambda^2}\int_{\big\{x\in\Omega, u_n<k\big\}} |(-\Delta)^{\frac{s}{2}}u_n|^2\,dx,$$

we apply Corollary 2.8 and we deduce that

$$\begin{aligned} &\text{meas }\big(\big\{x \in \Omega\,:\,|(-\Delta)^{\frac{s}{2}}u_n| \geqslant \lambda, u_n < k\big\}\big) \\ &\leqslant \frac{1}{\lambda^2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}T_k(u_n)|^2\,dx \leqslant \frac{k}{\lambda^2}\|f_n\|_{L^1(\Omega)}. \end{aligned}$$

Moreover, using (4.9), we have that for every $k > 0$,

$$\text{meas }\big(\big\{x \in \Omega\,:\,|(-\Delta)^{\frac{s}{2}}u_n| \geqslant \lambda\big\}\big) \leqslant \frac{k}{\lambda^2}\|f_n\|_{L^1(\Omega)} + c\Big(\frac{\|f_n\|_{L^1(\Omega)}}{k}\Big)^{\frac{N}{N-2s}}\,.$$

Minimizing in k we find that the minimum is achieved by $k = \lambda^{\frac{N-2s}{N-s}} \|f_n\|_{L^1(\Omega)}^{\frac{s}{N-s}}$, thus we have

$$\text{meas}\,\big(\big\{x \in \Omega \,:\, |(-\Delta)^{\frac{s}{2}} u_n| \geqslant \lambda\big\}\big) \leqslant c\Big(\frac{\|f_n\|_{L^1(\Omega)}}{\lambda}\Big)^{\frac{N}{N-s}}. \tag{4.11}$$

This means that $\big|(-\Delta)^{s/2} u_n\big|$ is bounded in the Marcinkiewicz space $\mathcal{M}^{\frac{N}{N-s}}(\Omega)$ and consequently (4.10) holds true.

Step 3. *Passing to the limit.*

Before passing to the limit in the equation, we need to determine the a.e. limit of u_n. Using the linearity of the equation, we have that for any m and $n \in \mathbb{N}$, then $u_n - u_m$ solves

$$\begin{cases} \mathcal{L}(u_n - u_m) = f_n - f_m & \text{in } \Omega, \\ u_n = 0\,, \quad u_m = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Hence, choosing for any $k > 0$, $T_k(u_n - u_m)$ as a test function in the weak formulation of the above problem, we deduce, by repeating the computations of Step 1, that

$$\text{meas}\,\big(\{x \in \Omega \,:\, |u_n - u_m| \geqslant k\}\big) \leqslant c\big(\frac{\|f_n - f_m\|_{L^1(\Omega)}}{k}\big)^{\frac{N}{N-2s}}.$$

Since the right hand side of the above inequality is small for n and m large enough, it follows that $\{u_n\}$ is a Cauchy sequence in measure. Consequently, up to subsequences (not relabeled), it converges in Ω almost every where, toward a function u.

By the Step 1 we also deduce (using the embedding of the $\mathcal{M}^p(\Omega)$ spaces into the $L^p(\Omega)$, for $p \geqslant 1$) that u_n also converges to u in $L^q(\Omega)$, for any $1 \leqslant q < \frac{N}{N-2s}$. Notice that this is sufficient to pass to the limit in the equation and obtain a weak solution of (4.1). Observe that, by the uniqueness, the whole sequence converges to u in $L^q(\Omega)$ and that (4.4) holds.

Since (4.11) holds, we have that

$$\text{meas}\,\big(\big\{x \in \Omega \,:\, |(-\Delta)^{\frac{s}{2}}(u_n - u_m)| \geqslant \lambda\big\}\big) \leqslant c\Big(\frac{\|f_n - f_m\|_{L^1(\Omega)}}{\lambda}\Big)^{\frac{N}{N-s}},$$

thus $(-\Delta)^{\frac{s}{2}}(u_n)$ is a Cauchy sequence in measure in Ω, therefore, up to a subsequence, $(-\Delta)^{\frac{s}{2}}(u_n)$ converges a.e in Ω. Hence by Fatou lemma (4.3) follows by (4.8). Again by Fatou lemma and (4.11), we also obtain (4.5). □

4.1. **Calderón-Zygmund type result for weak solutions.** In the local case it is well known that the Calderón-Zygmund results are true if the right hand side is $L^m(\Omega)$ for $1 < m < \infty$ (see [12], for instance). The existence of a unique weak solution of (3.1) is a consequence of Theorem 4.4. Thus we can define the operator T as in (3.2) and we can get the Calderón-Zygmund type results in such a range of m by using the interpolation Theorem 2.9.

The statement of our last result concerning solutions of elliptic problems is the following.

Theorem 4.7. *Let f belong to $L^m(\Omega)$, with $1 < m < \frac{2N}{N+2s}$. Then there exists a unique weak solution of (3.1). Moreover there exists a constant c, only depending on N, Ω, m and s, such that*

$$\|u\|_{L^{m_s^{**}}(\Omega)} \leqslant c\,\|f\|_{L^m(\Omega)} \tag{4.12}$$

and

$$\|(-\Delta)^{s/2}u\|_{L^{m^{**}_{s/2}}(\Omega)} \leqslant c\,\|f\|_{L^m(\Omega)} \tag{4.13}$$

where

$$\frac{1}{m_t^{**}} = \frac{1}{m} - \frac{2t}{N}, \qquad \forall t \in (0,1).$$

Proof. In order to get the estimate we argue by interpolation, namely we apply above interpolation theorem. We define the map T as the inverse of $\mathcal{L}$ (see (3.2)) and, due to Theorem 4.4 we know that it acts between $L^1(\Omega)$ and $L^r(\Omega)$, for any $r < \frac{N}{N-2s}$. On the other hand, estimate (4.12) in the case $(2_s^*)' = m$ tells us that it acts between $L^{(2_s^*)'}(\Omega)$ and $L^{2_s^*}(\Omega)$, too. Consequently

$$\|u_n\|_{L^p(\Omega)} = \|Tf_n\|_{L^p(\Omega)} \leqslant c\|f_n\|_{L^q(\Omega)}$$

where

$$\frac{1}{p} = \frac{1-\theta}{2_s^*} + \frac{\theta}{r} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\theta}{(2_s^*)'} + \theta,$$

i.e.

$$\frac{1}{q} - \frac{1}{p} = \frac{2s}{N} \qquad \text{that implies} \qquad \frac{1}{p} = \frac{1}{m_s^{**}}\,.$$

On the other hand, arguing in the same way with the norms of $|(-\Delta)^{s/2}u_n|$ we deduce (4.13).

□

5. Parabolic problem: Finite energy setting

Let consider the following parabolic problem:

$$\begin{cases} u_t + \mathcal{L}u = f(x,t) & \text{in } Q_T, \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega \times \{0\}, \end{cases} \tag{5.1}$$

where u_0 and f are functions defined in suitable Lebesgue spaces, $T > 0$ and Ω is a bounded subset of $\mathbb{R}^N$, $N \geqslant 2$, with Lipschitz boundary.

As in the elliptic case, we are interested on existence, uniqueness and summability of the solution u with respect to the summability of the datum f.

First, we define the meaning of a finite energy solution.

Definition 5.1. We say that $u \in L^2(0,T;H_0^s(\Omega)) \cap \mathcal{C}([0,T],L^2(\Omega))$ with $u_t \in L^2(0,T;H^{-s}(\Omega))$ is a finite energy solution for the parabolic problem

$$\begin{cases} u_t + \mathcal{L}u = f(x,t) & \text{in } Q_T, \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega \times \{0\}, \end{cases}$$

with $f \in L^2(0,T;H^{-s}(\Omega))$ and $u_0 \in L^2(\Omega)$ if it satisfies

$$\int_0^T \int_\Omega u_t w \, dx\, dt + \int_0^T \mathcal{E}\big(u(x,t), w(x,t)\big)\, dt = \int_0^T \langle f, w\rangle\, dt,$$

for any $w \in L^2(0,T;H_0^s(\Omega))$, where $\mathcal{E}\big(\cdot,\cdot\big)$ is the bilinear form defined in (1.4).

Remark 5.2. Observe that we ask u to belong to $\mathcal{C}([0,T],L^2(\Omega))$ in order to have the identity $u(x,0)=u_0(x)$ in the $L^2(\Omega)$ sense. Anyway, Theorem 2.15 guarantees that if $u\in L^2(0,T;H^s_0(\Omega))$ and $u_t\in L^2(0,T;H^{-s}(\Omega))$, then $u\in\mathcal{C}([0,T];L^2(\Omega))$.

The existence and uniqueness of an energy solution to the problem (5.1) for f in the dual space $L^2(0,T;H^{-s}(\Omega))$ and $u_0\in L^2(\Omega)$ can be proved by means of a direct abstract Hilbert space approach. For the reader convenience, we include here the proof inspired by the one of A.N. Milgram (see [36]) for the local case, which is based on a method by Vishik (see [54]).

Theorem 5.3. *Assume that $f\in L^2(0,T;H^{-s}(\Omega))$, then for any $u_0\in L^2(\Omega)$ problem (5.1) has a unique finite energy solution.*
Moreover if f is also a nonnegative function and $u_0\geqslant 0$, such a solution is nonnegative, too.

Proof. Let $\mathcal{C}^\infty_*(\Omega\times[0,T])$ denote the $\mathcal{C}^\infty(\Omega\times[0,T])$ functions that vanish in $(\mathbb{R}^N\setminus\Omega)\times[0,T]$ and in $\Omega\times\{T\}$. Consider $\phi\in\mathcal{C}^\infty_*(\Omega\times[0,T])$, $u\in L^2(0,T;H^s_0(\Omega))$, and define the operator

$$L_\phi(u):=\int_0^T\int_\Omega -u\phi_t\,dx\,dt+\int_0^T\mathcal{E}\big(u(x,t),\phi(x,t)\big)\,dt.$$

Notice that u is an energy solution to (P) with $f\in L^2(0,T;H^{-s}(\Omega))$ if and only if

$$L_\phi(u)=\int_0^T\langle f,\phi\rangle\,dt+\int_\Omega u(x,0)\phi(x,0)\,dx.$$

We also define the following inner product,

$$\langle\varphi,\phi\rangle_*=\frac{1}{2}\langle\varphi(x,0),\phi(x,0)\rangle_{L^2(\Omega)}+\int_0^T\mathcal{E}\big(\varphi(x,t),\phi(x,t)\big)\,dt, \tag{5.2}$$

and denote by $H^*(\Omega\times[0,T])$ the Hilbert space built as the completion of $\mathcal{C}^\infty_*(\Omega\times[0,T])$ with the norm $\|\phi\|_*$ induced by the inner product (5.2).
Observe that for any $\varphi\in L^2(0,T;H^s_0(\Omega))$, by Hölder and Sobolev inequalities, we have that

$$|L_\phi(\varphi)|\leqslant c_\phi\left(\|\varphi\|_{L^2(0,T,L^2(\Omega))}+\lambda^{-1}\|\varphi\|_{L^2(0,T,H^s_0(\Omega))}\right)\leqslant\tilde{c}_\phi\|\varphi\|_*.$$

Therefore, L_ϕ is a linear continuous functional in $H^*(\Omega\times[0,T])$, and by the Fréchet-Riesz Theorem, there exists $\mathcal{T}\phi\in H^*(\Omega\times[0,T])$ such that

$$L_\phi(\varphi)=\langle\varphi,\mathcal{T}\phi\rangle_*\quad\text{ for all }\varphi\in H^*(\Omega\times[0,T]).$$

Since $\mathcal{T}$ is a linear operator in $H^*(\Omega\times[0,T])$, then it is injective, too. Moreover

$$L_\phi(\phi)=\frac{1}{2}\int_\Omega\phi^2(x,0)\,dx+\int_0^T\mathcal{E}\big(\phi(x,t),\phi(x,t)\big)\,dt=\|\phi\|_*^2,$$

and consequently, $\langle\phi,\mathcal{T}\phi\rangle_*=\|\phi\|^2_*$. Thus, by the Cauchy-Schwartz inequality,

$$\|\phi\|^2_*\leqslant\|\phi\|_*\|\mathcal{T}\phi\|_*,\qquad\text{i.e.,}\qquad\|\phi\|_*\leqslant\|\mathcal{T}\phi\|_*.$$

Therefore, this implies that $\mathcal{T}$ is bijective and its inverse $\mathcal{T}^{-1}$ has a norm less than or equal to 1, and can be extended to the closure M of $Rank(\mathcal{T})$.

On the other hand, let us define

$$B_{u_0,f}(\phi):=\int_\Omega u_0\phi(x,0)\,dx+\int_0^T\int_\Omega\phi f\,dx\,dt.$$

Denoting $\phi_0 := \phi(x,0)$, by Hölder inequality we have that

$$|B_{u_0,f}(\phi)| \leqslant \|u_0\|_{L^2(\Omega)}\|\phi_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))}\|\phi\|_{L^2(0,T;L^2(\Omega))} \leqslant c_{u_0,f,\lambda}\|\phi\|_*,$$

and thus,

$$|B_{u_0,f}(\mathcal{T}^{-1}\psi)| \leqslant c\|\mathcal{T}^{-1}\psi\|_* \leqslant c\|\psi\|_*.$$

Therefore, by applying the Fréchet-Riesz Theorem again, there exists a unique $u \in M$ such that $B_{u_0,f}(\mathcal{T}^{-1}\psi) = \langle \psi, u\rangle_*$ for every $\psi \in M$. Calling $\phi = \mathcal{T}^{-1}\psi$ this means

$$B_{u_0,f}(\phi) = \langle \mathcal{T}\phi, u\rangle_* = L_\phi(u),$$

that is,

$$\int_0^T\int_\Omega -u\phi_t\,dx\,dt + \int_0^T \mathcal{E}\big(u(x,t),\phi(x,t)\big)\,dt = \int_0^T\int_\Omega f\,\phi\,dx\,dt + \int_\Omega u(x,0)\phi(x,0)\,dx,$$

where $\phi \in L^2(0,T;H_0^s(\Omega))$ and $\phi_t \in L^2(0,T;H^{-s}(\Omega))$. Finally, by a density argument, one can conclude, integrating by parts, that in fact $u \in L^2(0,T;H_0^s(\Omega))$, $u_t \in L^2(0,T;H^{-s}(\Omega))$, and

$$\int_0^T\int_\Omega u_t\phi\,dx\,dt + \int_0^T \mathcal{E}\big(u(x,t),\phi(x,t)\big)\,dt = \int_0^T\int_\Omega f\phi\,dx\,dt.$$

Thus $u(x,t)$ is an energy solution of (P). □

6. Nonvariational setting: weak solutions

We define the notion of weak solution just asking the regularity needed to give distributional sense to the equation.

Let us consider the class of test functions:

$$\mathcal{P}(Q_T) = \{\phi(\cdot,t) \in C^1([0,T),\mathcal{C}_0^\beta(\Omega)) \quad \text{is a solution to } (P)\}, \tag{6.1}$$

$$\text{where } (P) \equiv \begin{cases} -\phi_t + \mathcal{L}(\phi) = \psi & \text{in } Q_T, \\ \phi(x,t) = 0 & \text{in } (\mathbb{R}^N\setminus\Omega)\times[0,T), \\ \phi(x,T) = 0 & \text{in } \Omega\,, \end{cases}$$

with $\psi \in \mathcal{C}_0^\infty(Q_T)$.

Notice that the existence of a regular solution is guaranteed by the result by Felsinger and Kassmann in [27] (for a kernel $\mathcal{K}$ more general than ours).

Definition 6.1. We say that $u \in C([0,T];L^1(\Omega))$ is a *weak solution* to (5.1) for $f \in L^1(Q_T)$ if for all $\phi \in \mathcal{P}(Q_T)$, $\phi = 0$ on $(\mathbb{R}^n\setminus\Omega)\times[0,T)$ and $\phi(x,T) = 0$ in Ω, we have that

$$\int_{Q_T} u\psi\,dx\,dt = \int_{Q_T} f\phi\,dx\,dt + \int_\Omega u_0(x)\psi(x,0)\,dx\,.$$

Henceforth, let us introduce the following notations: for any measurable v we define

$$\begin{gathered} A_k(v) = \{(x,t)\in Q_T : v(x,t) > k\}, \\ \text{and for a.e. } t\in(0,T),\quad A_k^t(v) = \{x\in\Omega : v(x,t) > k\}. \end{gathered} \tag{6.2}$$

Theorem 6.2. *Let $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$. Then there exists a unique weak solution to (5.1). Moreover:*

$$\forall k \geqslant 0, \qquad T_k(u) \in L^2(0,T; H_0^s(\Omega))\,, \tag{6.3}$$

$$\begin{aligned} &u \in L^q(Q_T)\,,\ \forall q \in \Big(1, \tfrac{N+2s}{N}\Big) \\ \textit{and}\quad &\big|(-\Delta)^{\frac{s}{2}} u\big| \in L^r(Q_T)\,,\ \forall r \in \Big(1, \tfrac{N+2s}{N+s}\Big)\,. \end{aligned} \tag{6.4}$$

Remark 6.3. Observe that if $s = 1$ the above results correspond to the classical ones for second order parabolic problems (see for instance [11]).

Proof. UNIQUENESS.- Let w be a very weak solution of (5.1) with $f = 0$ and $u_0 = 0$, i.e.

$$\begin{cases} w_t + \mathcal{L}w = 0 & \text{in } Q_T, \\ w = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ w(x,0) = 0 & \text{in } \Omega\,; \end{cases}$$

we want to prove that $w \equiv 0$. Consider $F \in \mathcal{C}_0^\infty(Q_T)$, and let ϕ_F be the solution of the backward problem

$$\begin{cases} -(\phi_F)_t + \mathcal{L}\phi_F = F & \text{in } Q_T, \\ \phi_F = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ \phi_F(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{6.5}$$

Choosing ϕ_F as a test function we deduce that for any $F \in \mathcal{C}_0^\infty(Q_T)$,

$$\int_0^T \int_\Omega wF\,dx\,dt = 0,$$

that means, $w = 0$ in $\mathcal{D}'(Q_T)$.

EXISTENCE.- We obtain the solution as a limit of solutions to approximated problems. Let us consider $u_n \in L^2(0,T; H_0^s(\Omega)) \cap L^\infty(Q_T)$ the solution to the approximated problems

$$\begin{cases} (u_n)_t + \mathcal{L}u_n = f_n & \text{in } \Omega \times (0,T), \\ u_n(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u_n(x,0) = u_{n0}(x) & \text{in } \Omega, \end{cases} \tag{6.6}$$

where $f_n = T_n\big(f(x,t)\big)$ and $u_{0n} = T_n\big(u_0(x)\big)$ are smooth functions.

Step 1. *There exists a positive constant c, depending only on N, Ω and s, such that*

$$\|u_n\|_{L^q(Q_T)} \leqslant c\,\|f_n\|_{L^1(Q_T)}, \qquad \forall q \in \Big(1, \frac{N+2s}{N}\Big)\,. \tag{6.7}$$

We take $T_k(u_n)$ as a test function in (6.6) and, by a standard argument, we deduce that there exists a positive constant c, just depending on Ω and $\|f_n\|_{L^1(Q_T)}$, such that

$$\|u_n\|_{L^\infty(0,T,L^1(\Omega))} \leqslant c \qquad \text{and} \qquad \|T_k(u_n)\|_{L^2(0,T,H_0^s(\Omega))} \leqslant c\,k.$$

Observe moreover that by interpolation we have

$$\int_\Omega |T_k(u_n)|^\sigma\,dx \leqslant \Big(\int_\Omega |T_k(u_n)|\,dx\Big)^{(1-\theta)\sigma} \Big(\int_\Omega |T_k(u_n)|^{2_s^*}\,dx\Big)^{\frac{\sigma\theta}{2_s^*}},$$

with $\theta \in (0,1)$ such that $\frac{1}{\sigma} = 1-\theta+\frac{\theta}{2^*_s}$. This and Sobolev inequality yield to

$$\int_\Omega |T_k(u_n)|^\sigma\,dx \leqslant c\Big(\int_\Omega |T_k(u_n)|\,dx\Big)^{(1-\theta)\sigma}\Big(\int_\Omega |T_k(u_n)|\,dx + \|T_k(u_n)\|_{H^s_0(\Omega)}\Big)^{\sigma\theta},$$

for a $c>0$. Choosing $\theta\sigma = 2$, $\sigma = \frac{2(N+s)}{N}$, and integration over $(0,T)$, we get

$$\int_0^T\int_\Omega |T_k(u_n)|^\sigma\,dx\,dt \leqslant ck.$$

Therefore,

$$k^\sigma \text{ meas}\{(x,t)\in\Omega\times(0,T),\, |u_n(x,t)|\geqslant k\} \leqslant C_M k,$$

and then u_n is bounded in the Marcinkiewicz space $\mathcal{M}^{\frac{N+2s}{N}}(Q_T)$.

Step 2. *There exists a positive constant c, depending on q, N, Ω and s, such that*

$$\|(-\Delta)^{\frac{s}{2}}u_n\|_{L^q(Q_T)} \leqslant c\,\|f_n\|^q_{L^1(Q_T)}, \qquad \forall q\in\Big(1,\frac{N+2s}{N+s}\Big). \tag{6.8}$$

For any $\lambda>0$, we have that

$$\big\{(x,t)\in Q_T \,:\, |(-\Delta)^{\frac{s}{2}}u_n|\geqslant\lambda\big\}$$

$$=\big\{(x,t)\in Q_T \,:\, |(-\Delta)^{\frac{s}{2}}u_n|\geqslant\lambda, |u_n|<k\big\}\cup\big\{(x,t)\in Q_T \,:\, |(-\Delta)^{\frac{s}{2}}u_n|\geqslant\lambda, |u_n|\geqslant k\big\},$$

so that

$$\big\{(x,t)\in Q_T \,:\, |(-\Delta)^{\frac{s}{2}}u_n|\geqslant\lambda\big\}\subset\big\{(x,t)\in Q_T \,:\, |(-\Delta)^{\frac{s}{2}}u_n|\geqslant\lambda, |u_n|<k\big\}\cup A^t_k(u_n),$$

where $A^t_k(u_n)$ has been defined in (6.2). Since

$$\text{meas}\,\big(\{(x,t)\in Q_T \,:\, |(-\Delta)^{\frac{s}{2}}u_n|\geqslant\lambda, |u_n|<k\}\big)$$
$$\leqslant \frac{1}{\lambda^2}\int_0^T\int_{\{(x,t)\in\Omega_T, |u_n|<k\}} |(-\Delta)^{\frac{s}{2}}u_n|^2\,dx\,dt,$$

applying Corollary 2.8, we deduce that

$$\text{meas}\,(\{(x,t)\in Q_T \,:\, |(-\Delta)^{\frac{s}{2}}u_n|\geqslant\lambda, |u_n|<k\})$$
$$\leqslant \frac{1}{\lambda^2}\int_0^T\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}T_k(u_n)|^2\,dx \leqslant \frac{k}{\lambda^2}\|f_n\|_{L^1(Q_T)},$$

and then we have that for every $k>0$,

$$\text{meas}\,(\{(x,t)\in Q_T \,:\, |(-\Delta)^{\frac{s}{2}}u_n|\geqslant\lambda\}) \leqslant \frac{k}{\lambda^2}\|f_n\|_{L^1(Q_T)} + c\Big(\frac{\|f_n\|_{L^1(Q_T)}}{k}\Big)^{\frac{N+2s}{N}}.$$

Minimizing in k we find that the minimum is achieved in $k = \lambda^{\frac{N}{N+s}}\,\|f\|^{\frac{s}{N+s}}_{L^1(Q_T)}$, then the above inequality becomes

$$\text{meas}\,(\{|(-\Delta)^{\frac{s}{2}}u_n|\geqslant\lambda\}) \leqslant c\Big(\frac{\|f\|_{L^1(Q_T)}}{\lambda}\Big)^{\frac{N+2s}{N+s}}. \tag{6.9}$$

Step 3. *Passing to the limit.*

We have obtained in Step 1 that the sequence $\{u_n\}$ is bounded in the Marcinkiewicz space $\mathcal{M}^{\frac{N+2s}{N}}(Q_T)$. Using the linearity of the operator, we conclude that the sequence u_n a.e. converge to a function u in Ω, and by the embedding of $\mathcal{M}^{\frac{N+2s}{N}}(Q_T)$ in $L^q(Q_T)$, for any $q<\frac{N+2s}{N}$ and the uniqueness of the solution, the whole sequence u_n converges to u in $L^q(Q_T)$. Then, it follows that u is a weak solution to (5.1).

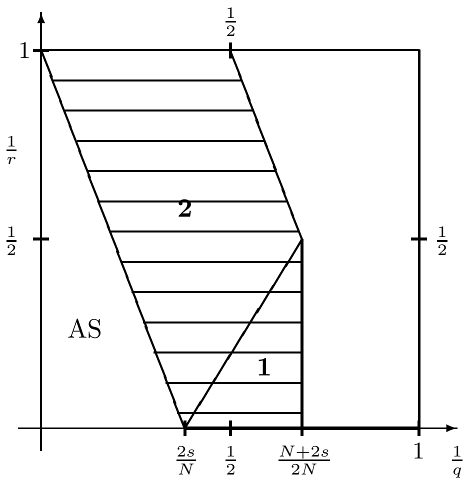
As in the elliptic case, from (6.9) and the linearity of the operator we conclude that the sequence $(-\Delta)^{\frac{s}{2}}u_n$ converge a.e. in Q_T to $(-\Delta)^{\frac{s}{2}}u$ and therefore by Step 2, using Fatou's Lemma, u verifies (6.4) $\square$

7. A priori estimates and summability of the solutions

In this section we deal with the summability of the solutions that we found in the previous section when the righthand side has extra integrability.

More precisely, we will study the optimal summability of the solution in terms of the summability of the data.

Let us represent the summability of the datum $f \in L^r(0,T;L^q(\Omega))$ in a diagram with axes $\dfrac{1}{q}$ and $\dfrac{1}{r}$. Since r, $q \in [1,+\infty]$, then all the possible cases of summability are inside of the square $[0,1]\times[0,1]$ (we use the notation $\dfrac{1}{\infty}=0$).



In this figure, the zone AS, called *Aronson-Serrin*, is delimitated by the line $\frac{1}{r}+\frac{N}{2qs}=1$. Parallel to this AS-line, in zone 2, we have the straight line $\frac{N}{2qs}+\frac{1}{r}=1+\frac{N}{4s}$ for $r<2$. The zones 1 and 2 are separated by the line $\frac{1}{r}=\frac{N}{N-2}\frac{1}{q}-\frac{2}{N-2}$ with $r>2$. Finally, the zone 1 is delimited on the right by the line $q=\frac{2N}{N+2s}$.

Figure 1

7.1. **Bounded solutions.** The zone AS, called *Aronson-Serrin*, corresponds to the set of data for which we find bounded solutions; in the case of second order differential operators (as, for instance, the Laplacian) such a result was obtained by D.G. Aronson and J.Serrin in [7]. The solutions for zones 1 and 2 are not expected to be bounded, but we can determine their summability in some Lebesgue space (for $s=1$, the local case, we recover the regularity results in [13]).

Let us consider the following approximated problems,

$$\begin{cases} u_{nt}+\mathcal{L}u_n=f_n(x,t) & \text{in } Q_T,\\ u_n(x,t)=0 & \text{in } (\mathbb{R}^N\setminus\Omega)\times[0,T),\\ u_n(x,0)=u_{n0}(x) & \text{in } \Omega\times\{0\}, \end{cases} \tag{7.1}$$

where $f_n(x,t)=T_n(f(x,t))$ and $f\geqslant 0$ in a suitable Lebesgue space.
Observe that, since f_n is bounded,the existence of a sequence $\{u_n\}$ of energy solutions to (7.1) is guaranteed by Theorem 5.3.

The main result in this paragraph is the L^∞ estimate for u_n.

Theorem 7.1. *Assume* $f \in L^r(0,T;L^q(\Omega))$ *with* r, q *satisfying*

$$\frac{1}{r} + \frac{N}{2qs} < 1 \tag{7.2}$$

and suppose that $u_{n0}(x) \in L^\infty(\Omega)$. *Then there exists a positive constant* c *such that the unique finite energy solution of (7.1) satisfies*

$$\|u_n(x,t)\|_{L^\infty(Q_T)} \leqslant c\|f\|_{L^r(0,T;L^q(\Omega))}.$$

Proof. According to the classical proof by D.G. Aronson and J. Serrin, the idea is to prove an L^∞ bound for u_n in $\Omega \times (0,\tau)$, for a positive (small) τ (to be fixed), and then to iterate such an estimate. Hence, we consider the energy formulation for (7.1) in Q_τ and we choose there $G_k(u_n)$ as a test function; hence we get

$$\begin{aligned}\int_\Omega \varphi_k(u_n(x,\tau))\,dx + \int_0^\tau \mathcal{E}\big(u_n(x,t), G_k(u_n(x,t))\big)\,dt\\ = \int_0^\tau\int_\Omega f_n G_k(u_n)\,dx\,dt + \int_\Omega \varphi_k(u_n(x,0))\,dx,\end{aligned}$$

where $\varphi_k(s) = \displaystyle\int_0^s G_k(\sigma)\,d\sigma$.

Consider a positive k large enough (let us say $k \geqslant \|u_{n0}\|_{L^\infty(\Omega)}$, in order to neglect the last term above), we deduce that

$$\|G_k(u_n)\|^2_{L^\infty(0,\tau;L^2(\Omega))} + \|G_k(u_n)\|^2_{L^2(0,\tau;H_0^s(\Omega))} \leqslant c\int_0^\tau\int_{A_k^t} f\,G_k(u_n)\,dx\,dt.$$

Notice that the term of the right hand side above can be estimated as follows,

$$\int_0^\tau\int_{A_k^t} f\,G_k(u_n)\,dx\,dt \leqslant \int_0^\tau\int_{A_k^t} f\,G_k(u_n)^2\,dx\,dt + \int_0^\tau\int_{A_k^t} f\,dx\,dt. \tag{7.3}$$

Let us study each member of the previous inequality. First, we define, for any $\chi \in (0,1)$

$$\bar r = 2r',\ \bar q = 2q',\ \hat r = \bar r(1+\chi),\ \hat q = \bar q(1+\chi),\ \chi = \frac{2\chi_1}{N}, \tag{7.4}$$

where $\chi_1 = 1 - \frac{1}{r} - \frac{N}{2qs}$ that is (strictly) positive thanks to (7.2). Consequently, $\dfrac{1}{\hat r} + \dfrac{N}{2\hat q s} = \dfrac{N}{4s}$. Thus by Hölder and Gagliardo-Nirenberg inequality (see (2.9)), we deduce that

$$\begin{aligned}\int_0^\tau\int_{A_k^t} f\,G_k(u_n)^2\,dx\,dt &\leqslant c\mu(k)^{\frac{2\chi}{\hat r}}\ \|G_k(u_n)\|^2_{L^{\hat r}(0,\tau;L^{\hat q}(A_k^t))}\\ &\leqslant c\mu(k)^{\frac{2\chi}{\hat r}}\left(\int_0^\tau \|G_k(u_n)\|^{(1-\theta)\hat r}_{L^2(\Omega)}\ \|(-\Delta)^{s/2}G_k(u_n)\|^{\hat r\theta}_{L^2(\mathbb{R}^N)}\,dt\right)^{\frac{2}{\hat r}}\\ &\leqslant c\mu(k)^{\frac{2\chi}{\hat r}}\left[\|G_k(u_n)\|^2_{L^\infty(0,\tau;L^2(\Omega))} + \|(-\Delta)^{s/2}G_k(u_n)\|^2_{L^2(0,\tau;L^2(\mathbb{R}^N))}\right]\end{aligned}$$

where $\mu(k) = \int_0^\tau |A_k^t|^{\frac{\hat r}{\hat q}}\,dt$ we also applied Young inequality and (7.4) (i.e. $\hat r\theta = 2$).

On the other hand the second term on the right hand side in (7.3) can be estimated by means of Hölder inequality, so that

$$\int_0^\tau\int_{A_k^t} f\,dx\,dt\ \leqslant c\mu(k)^{\frac{2(1+\chi)}{\hat r}}.$$

Denoting

$$|||G_k(u_n)|||^2 = \|G_k(u_n)\|^2_{L^\infty(0,\tau;L^2(\Omega))} + \|(-\Delta)^{s/2}G_k(u_n)\|^2_{L^2(0,\tau;L^2(\mathbb{R}^N))},$$

we deduce that

$$|||G_k(u_n)|||^2 \leqslant c\big[\mu(k)^{\frac{2\chi}{\hat r}}|||G_k(u_n)|||^2 \ + \mu(k)^{\frac{2(1+\chi)}{\hat r}}\big].$$

Notice that $\mu(\tau) \leqslant \tau|\Omega|$, so that we can fix τ, independent of u, suitable small in such a way that by using again the Gagliardo-Nirenberg inequality, we deduce that

$$\|G_k(u_n)\|^2_{L^{\hat r}(0,\tau;L^{\hat q}(\Omega))} \leqslant c\mu(k)^{\frac{2(1+\chi)}{\hat r}}.$$

Consider $h > k > 0$. Then $A_h \subset A_k$, where A_h is as in (6.2), and

$$\mu(h) \leqslant \frac{c}{(h-k)^{\hat r}}\mu(k)^{1+\chi}.$$

Applying Lemma 3.6, we conclude that there exist a constant d, depending only on q, r, $\|f\|_{L^r(0,T;L^q(\Omega))}$ and s, such that $\mu(d) = 0$, that is

$$\|u_n\|_{L^\infty(\Omega\times[0,\tau])} \leqslant d.$$

Iterating this procedure in the sets $\Omega\times[\tau,2\tau],\cdots,\Omega\times[j\tau,T]$, where $T - j\tau \leqslant \tau$ we can conclude that

$$\|u_n\|_{L^\infty(Q_T)} \leqslant C \quad \text{uniformly in } n \in \mathbb{N}.$$

□

There are several interesting consequences of the above estimate; here we stress that it directly implies existence of a solution, with the definition of solution changing according to the summability of f.

Corollary 7.2. *Let f be a function $f \in L^r(0,T;L^q(\Omega))$ with r, q satisfying*

$$\frac{1}{r} + \frac{N}{2qs} < 1. \tag{7.5}$$

Then, the solution obtained as limit of the solutions to (7.1) *satisfies*

(1) $\|u\|_{L^\infty(Q_T)} \leqslant c$,
(2) $\|u\|_{L^2(0,T;H_0^s(\Omega))} \leqslant c$.
(3) *Moreover,*
  (a) *if $r \geqslant 2$ then $u_t \in L^2(0,T;H^{-s}(\mathbb{R}^N))$, and u is an energy solution;*
  (b) *If $1 < r < 2$ then the equation is satisfied in the following weak sense*

$$-\int_{Q_T} u\phi_t dxdt + \int_0^T \mathcal{E}(u,\phi)dt = \int_{Q_T} f\phi\, dxdt + \int_\Omega u(x,0)\phi(x,0)dx,$$

for any $\phi \in L^2(0,T;H_0^s(\Omega))\cap L^\infty(Q_T)\cap\mathcal{C}^1([0,T);L^1(\Omega))$, with $\phi(x,T) = 0$.

Proof. We first observe that since f_n are increasing, by the weak comparison principle (using that the equation is linear) we deduce that the sequence u_n is increasing, too. Thus, the uniform estimate gives 1). Moreover, testing the equations (7.1) with u_n, 2) easily follows.

Noticing that if $r \geqslant 2$ then $f \in L^2(0,T;L^{(2_s^*)'}(\Omega))$ we obtain $3,a)$. Finally $3,b)$ is obtained by a simple integration by parts argument in (7.1) and passing to the limit.

□

Going further into the summability of the solution, we consider the borderline case in which $\frac{1}{r}+\frac{N}{2qs}=1$. Having in mind what happens in the elliptic case, we prove that solutions are not anymore bounded, but have an exponential summability. Namely, we prove the following result.

Theorem 7.3. *Assume that f belongs to $L^r(0,T;L^q(\Omega))$, with $r\geqslant 1$, $q>\frac{N}{2s}$ such that*

$$\frac{1}{r}+\frac{N}{2qs}=1\,, \tag{7.6}$$

and let u_n be the solution of (7.1) with $u_{n0}(x)\equiv 0$. Then there exists an $\alpha>0$ such that

$$\|e^{\alpha u_n}\|_{L^\infty(0,T;L^2(\Omega))}\leqslant c\|f\|_{L^r(0,T;L^q(\Omega))}. \tag{7.7}$$

Consequently, u_n is bounded in $L^p(Q_T)$, for every $p\geqslant 1$.

Proof. We only sketch the proof, since it is very similar to the elliptic one that we have proved in Section 3 (see Theorem 3.8).

Consider the following auxiliary convex function

$$\Phi_k(\sigma)=\begin{cases} e^{\alpha\sigma}-1, & \text{if} \quad 0\leqslant\sigma<k\\ \alpha e^{\alpha k}(\sigma-k)+e^{\alpha k}-1, & \text{if} \quad \sigma\geqslant k\end{cases}$$

for $k>0$ and for some α to be fixed later. Thus we have that

$$\frac{1}{2}\|\Phi_k(u_n)\|^2_{L^\infty(0,T;L^2(\Omega))}+\lambda\|u_n\|^2_{L^2(0,T;H^s_0(\Omega))}\leqslant\int_0^T\int_\Omega\Phi_k'(u_n)\Phi_k(u_n)f\,dx\,dt.$$

Now, observe that for $0\leqslant u_n\leqslant k$ we have $\Phi_k'(u_n)=\alpha\left(e^{\alpha u_n}-1+1\right)=\alpha\Phi_k(u_n)+\alpha$. Hence,

$$\int_0^T\int_\Omega\Phi_k'(u_n)\Phi_k(u_n)f\,dx\,dt\leqslant$$

$$\alpha\int_0^T\int_{\{u_n\leqslant k\}}\Phi_k(u_n)^2\,f+\alpha\int_0^T\int_{\{u_n\leqslant k\}}\Phi_k(u_n)\,f+\alpha\,e^{\alpha k}\int_0^T\int_{\{u_n>k\}}\Phi_k(u_n)\,f\,.$$

From now on, we follow the same ideas of the elliptic case and we get (7.7). □

7.2. **Summability of the solutions outside of the Aronson-Serrin zone.** Outside the zone AS, the solutions are not expected to be bounded. We obtain the following summability results for the zones 1 and 2 of the Figure 1.

Theorem 7.4. *Assume $u_{n0}(x)\equiv 0$ and $f\in L^r(0,T;L^q(\Omega))$ with $r>1$, $q>1$, satisfying*

$$1<\tfrac{1}{r}+\tfrac{N}{2qs}\leqslant 1+\tfrac{N}{4s} \qquad \textit{and} \qquad \tfrac{1}{q}\leqslant\tfrac{1}{2}+\tfrac{s}{N} \quad \textit{(zones 1 and 2 in Figure 1)},$$

then there exists a positive constant c such that the sequence of finite energy solutions of (7.1) satisfies

$$\|u_n(x,t)\|_{L^\infty(0,T;L^{2\gamma}(\Omega))}+\|u_n(x,t)\|_{L^{2\gamma}(0,T;L^{2^*_s\gamma})}\leqslant c\|f\|_{L^r(0,T;L^q(\Omega))}\,,$$

where

$$\gamma=\begin{cases}\dfrac{q(N-2s)}{2(N-2qs)} & \text{if } \frac{1}{r}<\frac{N}{N-2}\frac{1}{q}-\frac{2}{N-2} \qquad (\text{zone 1 in Figure 1}),\\ \dfrac{qrN}{2(Nr-2qs(r-1))} & \text{if } \frac{1}{r}\geqslant\frac{N}{N-2}\frac{1}{q}-\frac{2}{N-2} \qquad (\text{zone 2 in Figure 1}).\end{cases} \tag{7.8}$$

Proof. Let us consider a positive Lipschitz convex function $\Phi(s)$, with $\Phi(0)=0$. We point out that, since u_n is a bounded energy solution, then $\Phi(u_n)$ belongs to the energy space, and then we are allowed to choose it as a test function in problem (7.1). Thus by (2.7), we deduce that

$$\int_0^T\int_\Omega \Phi(u_n)\Phi(u_n)_t\,dx\,dt+\lambda\int_0^T\|\Phi(u_n)\|^2_{H^s_0(\Omega)}\,dt\leqslant\int_0^T\int_\Omega\Phi'(u_n)\Phi(u_n)f\,dx\,dt. \tag{7.9}$$

Let us apply such an inequality with the choice $\Phi(s)=s^\gamma$, with $\gamma\geqslant 1$ as in (7.8). Thus applying Hölder and Sobolev inequalities, we obtain

$$\begin{aligned}&\frac{1}{2\gamma}\sup_{t\in[0,T]}\int_\Omega u_n^{2\gamma}\,dx+\lambda\int_0^T\|u_n^\gamma\|^2_{L^{2^*_s}(\Omega)}\,dt\\ &\leqslant\gamma\|f_n\|_{L^r(0,T;L^q(\Omega))}\left(\int_0^T\left[\int_\Omega u_n^{(2\gamma-1)q'}\,dx\right]^{\frac{r'}{q'}}dt\right)^{\frac{1}{r'}}.\end{aligned} \tag{7.10}$$

Denoting by

$$A=\left(\int_0^T\|u_n\|^{(2\gamma-1)r'}_{L^{(2\gamma-1)q'}(\Omega)}\,dt\right)^{\frac{1}{r'}},$$

and thanks to Sobolev's inequality, we get

$$\sup_{[0,T]}\int_\Omega u_n^{2\gamma+1}\,dx\leqslant 2\gamma\|f_n\|_{L^r(0,T;L^q(\Omega))}A,$$

and

$$\int_0^T\left[\int_\Omega u_n^{\gamma 2^*_s}\,dx\right]^{\frac{2}{2^*_s}}dt\leqslant\frac{\mathcal{S}^2}{\lambda}\gamma\|f_n\|_{L^r(0,T;L^q(\Omega))}A.$$

We distinguish now two cases depending on the size of q.

First case: $1<q<\dfrac{Nr}{N+2s(r-1)}$, i.e. zone 1 in Figure 1.

Observe that $1<q<\dfrac{Nr}{N+2s(r-1)}$ implies $q<r$ and $\dfrac{r'}{q'}<\dfrac{2}{2^*_s}$. Hence, we apply Hölder's inequality to estimate A and we get

$$A\leqslant T^{\frac{1}{r'}-\frac{2^*_s}{2q'}}\left[\int_0^T\left(\int_\Omega u_n^{(2\gamma-1)q'}\,dx\right)^{\frac{2}{2^*_s}}dt\right]^{\frac{2^*_s}{2q'}}.$$

Thus by Sobolev inequality (see (2.3)), we deduce that

$$\int_0^T\left[\int_\Omega u_n^{2^*_s\gamma}\,dx\right]^{\frac{2}{2^*_s}}dt\leqslant c\left[\int_0^T\left(\int_\Omega u_n^{(2\gamma-1)q'}\,dx\right)^{\frac{2}{2^*_s}}dt\right]^{\frac{2^*_s}{2q'}}.$$

Choose $2_s^*\gamma = (2\gamma-1)q'$, that is, $\gamma = \frac{1}{2}\cdot\frac{q(N-2s)}{(N-2qs)}$. Notice that since $q < \frac{N}{2s}$ it follows that $\frac{2_s^*}{2q'} < 1$. Thus we get

$$\int_0^T \left(\int_\Omega u_n^{(2\gamma_1-1)q'}dx\right)^{\frac{2}{2_s^*}} dt \leqslant c,$$

and therefore, $\|u_n\|_{L^r(0,T;L^{q_s^{**}}(\Omega))} \leqslant c$, with $r = \frac{q(N-2s)}{(N-2qs)} > q$.

Second case: $\frac{Nr}{N+2(r-1)} \leqslant q < \frac{N}{2s}r'$, i.e. zone 2 in Figure 1.
In this case we choose $\gamma = \frac{1}{2}\frac{qrN}{Nr-2qs(r-1)}$. Thus by interpolation, we get that $\frac{1}{(2\gamma-1)q'} = \frac{1-\theta}{2\gamma} + \frac{\theta}{2_s^*\gamma}$ and consequently

$$\int_0^T \|u_n\|^{r'(2\gamma-1)}_{L^{(2\gamma-1)q'}(\Omega)}\,dt \leqslant c\|u_n\|^{2\gamma\mu_1}_{L^\infty(0,T;L^{2\gamma}(\Omega))}\int_0^T \left(\int_\Omega u_n^{2_s^*\gamma}\,dx\right)^{\frac{2}{2_s^*}\mu_2} dt,$$

where $\mu_1 = \frac{(1-\theta)r'(2\gamma-1)}{2\gamma}$ and $\quad \mu_2 = \frac{\theta r'(2\gamma-1)}{2\gamma}$. Using (7.8), we have that

$$\int_0^T \|u_n\|^{r'(2\gamma-1)}_{L^{(2\gamma-1)q'}}\,dt \leqslant c\|u_n\|^{2\gamma\mu_1}_{L^\infty(0,T;L^{2\gamma}(\Omega))}\left(\int_0^T \left[\int_\Omega u_n^{2_s^*\gamma}\,dx\right]^{\frac{2}{2_s^*}} dt\right)^{\mu_2},$$

and since $\mu_2 \leqslant 1$, matching the above inequality with (7.10) we obtain

$$\int_0^T \left(\int_\Omega u_n^{(2\gamma-1)q'}dx\right)^{\frac{r'}{q'}} dt \leqslant c.$$

□

Remark 7.5. Under the assumptions of Theorem 7.4, if u is the energy solution of (5.1), then it belongs also to the following spaces

$$L^p(0,T;L^r(\Omega)), \quad p = \frac{4\gamma ms}{N(m-2)}, \quad r = \gamma m, \quad \forall\, m \in (2, 2_s^*], \tag{7.11}$$

where γ is the number defined in Theorem 7.4. Moreover uniform estimates hold true in the previous spaces.
In particular, choosing $m = 2_s^*$, we get the best summability exponent in the spacial variable (i.e. $2_s^*\gamma$) together with the lower summability exponent in the time variable, (i.e. 2γ) and we obtain

$$u \in L^{2\gamma}(0,T;L^{2_s^*\gamma}(\Omega))\,.$$

On the other hand, choosing $m = 2\frac{N+2s}{N}$ we get $p = r$ and hence

$$u \in L^\mu(Q), \qquad \mu = 2\gamma\frac{N+2s}{N}, \tag{7.12}$$

where γ is defined in (7.8).

7.3. **Further summability results.** In order to conclude our study of the summability of the solutions, we argue as follows: first we find the summability of the solutions with data in $L^\infty(0,T;L^q(\Omega))$, for $q \in [1, N/2s)$, i.e. the bottom line of Figure 1.

Once we found that the unique solution of problem (7.1) is bounded in the Lebesgue space $L^\infty(0,T;L^{q_s^{**}}(\Omega))$, we argue by interpolation. Indeed, assume to know, for any pairs of exponents (r_1, q_1) and (r_2, q_2), that the solutions u_n of (7.1) are bounded in some Lebesgue space. Then, by Theorem 2.12, we know the summability of the sequence $\{u_n\}$ for any r and q that satisfy (2.10). Roughly speaking, in Figure 1, given any pairs of coordinates, we can deduce the summability of u_n for any r and q that lay in the segment that joins $(\frac{1}{r_1}, \frac{1}{q_1})$ and $(\frac{1}{r_2}, \frac{1}{q_2})$.

Consequently, if we know the behavior of u_n in a subset $\mathcal{A}$ of the square $(0,1) \times (0,1)$ of Figure 1, we know it in its convex envelop $\mathcal{A}^*$. Thus, if we deduce the summability of u_n in the bottom line of Figure 1, i.e. if such a line belongs to $\mathcal{A}$, then $\mathcal{A}^* \supset (0,1) \times (0,1)$.

We deal, now, with the bottom segment

$$r = \infty \qquad \text{and} \qquad 1 < q < \frac{N}{2s}\,.$$

Theorem 7.6. *If u_n is a solution to* (7.1)*, with $f(x,t) \in L^\infty(0,T;L^q(\Omega))$ and $u_{n0}(x) \equiv 0$. Then*

i) *if $q > \frac{N}{2s}$, then $\|u_n\|_{L^\infty(Q_T)} \leqslant c$;*

ii) *if $1 < q < \frac{N}{2s}$, then $\|u_n\|_{L^\infty(0,T;L^{q_s^{**}}(\Omega))} \leqslant c$;*

iii) *if $q = 1$, then $\|u_n\|_{L^\infty(0,T;L^\sigma(\Omega))} \leqslant c$, $\forall \sigma \in \left(1, \frac{N}{N-2s}\right)$.*

Proof of Theorem 7.6. In order to prove Theorem 7.6, first we consider a preliminary result:
If v is a positive finite energy solution to

$$\begin{cases} v_t + \mathcal{L}v = F(x) & \text{in } Q_T, \\ v(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ v(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{7.13}$$

with $F \in L^{\frac{2N}{N+2s}}(\Omega)$, $F(x) \geqslant 0$, then $v_t(x,t) \geqslant 0$ for all (x,t) in Q_T.

Applying (weak) maximum principle (i.e. multiplying the equation by $v_n^-(x,t)$, and integrating by parts) , we conclude that $v(x,t) \geqslant 0$, for every (x,t) in Q_T. It is sufficient to prove that $v(x,t) \leqslant v(x,t+\tau)$, for all $\tau \in [0, T-t]$. This immediately follows thanks to the Comparison Principle for finite energy solutions, since, for $\tau \in [0, T-t]$,

$$\begin{aligned} v_t(x,t) + \mathcal{L}v(x,t) = F(x) &= v_t(x,t+\tau) + \mathcal{L}v(x,t+\tau), \\ v(x,0) = 0 &\leqslant v(x,t). \end{aligned}$$

Consequently, u_n are sub solution of the elliptic problem associated to (7.1): and applying the results of the elliptic sections, the conclusions follow. □

8. Elliptic and parabolic Kato inequalities

In the folklore of the area it is implicitly known the extension of Kato's inequality to the fractional Laplacian. For the sake of completeness, we include next a proof of Kato's inequality for weak solutions.

Theorem 8.1. *If $f \in L^1(\Omega)$ and $u \in H^s(\mathbb{R}^N)$ is a weak solution to*

$$\mathcal{L}u(x) = f(x),$$

then

$$\mathcal{L}(|u|)(x) \leqslant f(x)\,\mathrm{sign}(u) \quad \text{in the weak sense.} \tag{8.1}$$

That is, we have

$$\int_{\mathbb{R}^N} |u|\mathcal{L}\psi(x)dx \leqslant \int_{\mathbb{R}^N} \mathrm{sign}(u)f(x)\psi(x)dx, \quad \forall \psi \in \mathcal{C}_0^\infty, \quad \psi \geqslant 0.$$

Proof. Consider for $\epsilon \in (0,1)$ the regularization of $|s|$ given by

$$\varphi_\epsilon(s) = (|s|^2 + \epsilon^2)^{\frac{1}{2}} - \epsilon.$$

It is easy to verify that,

i) $\varphi_\epsilon(s) \to |s|$ uniformly as $\epsilon \to 0$,
ii) $\varphi'_\epsilon(s)$ is bounded uniformly with respect to ϵ.
iii) $\varphi_\epsilon(s)$ is convex.

If we assume that u is smooth enough, since φ_ϵ is convex and Lipschitz and satisfies $\varphi_\epsilon(0) = 0$, then

$$\begin{aligned}\mathcal{L}\varphi_\epsilon(u(x)) &= \int_{\mathbb{R}^N} \big(\varphi_\epsilon(u(x)) - \varphi_\epsilon(u(y))\big)k(x,y)\,dy \\ &\leqslant \int_{\mathbb{R}^N} \varphi'_\epsilon(u(x))\big(u(x)-u(y)\big)k(x,y)\,dy = \varphi'_\epsilon(u(x))\mathcal{L}u(x).\end{aligned}$$

Now if $\psi \in \mathcal{C}_0^\infty$, $\psi \geqslant 0$, we have,

$$\begin{aligned}&\int_{\mathbb{R}^N} \varphi_\epsilon(u(x))\mathcal{L}\psi(x)dx = \int_{\mathbb{R}^N} \mathcal{L}\varphi_\epsilon(u(x))\psi(x)dx \\ \leqslant &\int_{\mathbb{R}^N} \varphi'_\epsilon(u(x))\mathcal{L}u(x)\psi(x)dx = \int_{\mathbb{R}^N} \varphi'_\epsilon(u(x))f(x)\psi(x)dx.\end{aligned}$$

Passing to the limit with respect to ε in the previous inequality we obtain,

$$\int_{\mathbb{R}^N} |u|\mathcal{L}\psi(x)dx \leqslant \int_{\mathbb{R}^N} \mathrm{sign}(u)f(x)\psi(x)dx,$$

as desired.

□

Obviously, we have also a parabolic counterpart of such an inequality. We state it here for completeness, but we only sketch the proof since it follows with minor changes by the previous one.

Theorem 8.2 (Parabolic Kato's Inequality)**.** *Let Ω be a bounded domain in $\mathbb{R}^N$ and $p : \mathbb{R} \to \mathbb{R}$ a non decreasing bounded continuous function (except at most in a number of finite points). Then for any weak solution to*

$$u_t + \mathcal{L}u = f(x,t) \qquad \text{in } Q_T$$

with $f \in L^1(Q_T)$, we have that

$$\frac{\partial \Phi(u)}{\partial t} + \mathcal{L}\Phi(u) \leqslant \Big(\frac{\partial u}{\partial t} + \mathcal{L}u\Big)\varphi(u) \quad \text{in the weak sense,}$$

where

$$\Phi(s) = \int_0^s \varphi(\sigma)\,d\sigma.$$

Remark 8.3. Observe that applying the above result with the choice $\varphi(s) = \text{sign}\,(s)$, we get

$$\frac{\partial |u|}{\partial t} + \mathcal{L}|u| \leqslant \Big(\frac{\partial u}{\partial t} + \mathcal{L}u\Big)\ \text{sign}\,(u) \quad \text{ in the weak sense,}$$

while for $\varphi(u) = \text{sign}^{\,+}(u)$, we deduce

$$\frac{\partial u^+}{\partial t} + \mathcal{L}u^+ \leqslant \Big(\frac{\partial u}{\partial t} + \mathcal{L}u\Big)\ \text{sign}^{\,+}(u) \quad \text{ in the weak sense.}$$

Proof. If $\varphi(s)$ is smooth, then the proof is direct. Otherwise we argue by approximation and we follow the same line as in the elliptic case. □

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