COMPARISON PRINCIPLES FOR p-LAPLACE EQUATIONS WITH LOWER ORDER TERMS

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Abstract. We prove comparison principles for quasilinear elliptic equations whose simplest model is

$$\lambda u - \Delta_p u + H(x, Du) = 0 \qquad x \in \Omega,$$

where $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$ is the p-Laplace operator with $p > 2$, $\lambda \geq 0$, $H(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function and $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 2$.

We collect several comparison results for weak sub and super solutions under different setting of assumptions and with possibly different methods. A strong comparison result is also proved for more regular solutions.

1. Introduction

In this paper we consider quasilinear elliptic equations of the type

$$\lambda u - \Delta_p u + H(x, Du) = 0 \qquad \text{in } \Omega, \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, where $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$ is the p-Laplace operator on a scalar function $u : \Omega \to \mathbb{R}$, λ is a nonnegative real number and $H(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function (i.e. measurable in x and continuous in ξ).

The purpose of this article is to give different sets of conditions under which the comparison principle holds for weak solutions, namely that given u and v a sub solution and a super solution, respectively, of the equation (1.1), if (in weak sense) $u \leq v$ at the boundary $\partial\Omega$, then $u \leq v$ in Ω. A strong comparison principle will also be proved under suitable assumptions.

In our study, we will be only concerned with the case $p > 2$, which makes the comparison principle more difficult since the operator turns out to be degenerate elliptic. However we stress that, even when $p = 2$, while for sufficiently smooth solutions the comparison follows from the maximum principle, for weak solutions the comparison principle is not trivial.

Let us first recall what is already known in the literature. In the case $p = 2$, general structure conditions on the lower order terms were given by Barles and Murat [6] to ensure that the comparison principle holds whenever H has the so-called natural growth with respect to Du (i.e. at most quadratic at infinity) and solutions belong to $H^1(\Omega) \cap L^\infty(\Omega)$. Their method relies on two steps; they first investigate under which conditions the comparison would follow from the natural linearization of H (and eventually of the operator, should it be a nonlinear divergence form operator in H^1), secondly they use some change of unknown to reduce the problem to the former good situation. This idea was a bit refined in [4] in order to deal with limiting cases of applicability, including some cases of unbounded solutions but with exponential integrability.

On the other hand, the comparison principle for *unbounded* solutions must be handled with care, and easy counterexamples show that it may fail. This is essentially due to the fact that the linearization of H needs a proper functional setting to work, therefore a sufficient integrability on the gradient of solutions. This problem only occurs if the growth of the Hamiltonian is sufficiently

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high, indeed for linear growth uniqueness results hold in more generality (see for instance [8], [9]). The problem of comparison and uniqueness for unbounded solutions in the supercritical cases was dealt with in [7] for $p = 2$ (and in [22] for $p \neq 2$ under more restrictive conditions) with an approach which relies on the convexity of H (with respect to Du) whenever the linearization argument should not work.

Different type of uniqueness results were also proved in [2], [14] for general nonlinear problems which include equation (1.1) but only if $1 < p < 2$. Indeed, to our knowledge there are not general results in the literature giving comparison principles for weak solutions to (1.1) when $p > 2$. Apart from some model cases contained in [22], some results were obtained when $H(x, Du)$ has precisely the growth as $|Du|^p$, in which case other methods can be used relying on the classical Hopf-Cole exponential transform, see e.g. [1], [3].

In this article we aim at giving rather general structure conditions on the function H which imply the comparison principle for weak solutions to (1.1) when $p > 2$ and in the possible different cases that $\lambda > 0$ or $\lambda = 0$. To this purpose we use alternatively either the approach of Barles and Murat, based on a linearization principle coupled with change of variables, or an approach mostly relying on the convexity of H. Thirdly, we will also exploit one more idea, consisting in the linearization of the operator. This method relies on the fundamental work [10] which suggests the use of weighted estimates on $|Du|^{p-2}$ in order for a useful linearization approach to work. We will use this latter method as a key-point in order to get a *strong* comparison principle.

We confine ourselves to the case of bounded solutions, in order not to mix different issues; however we point out that some results, namely those which only use the convexity properties of the function H, apply possibly to the study of unbounded solutions. A significant point that we wish to stress in our results is the role of the zero order term λu, by specifying how the comparison principle would hold under more general conditions whenever $\lambda > 0$. Our specific interest in this question is clearly motivated by the study of the parabolic problem and its long time behavior, a reason which we feel sufficiently strong for us to choose the linear term λu as our reference zero order term. In addition, the different homogenous scaling of this term allows us to observe interesting phenomena, compared with other possible choices of zero order terms as the homogeneous preserving growth $|u|^{p-2}u$. As an example, for the model problem

$$\lambda u - \Delta_p u + |Du|^q = f(x) \qquad \text{in } \Omega \tag{1.2}$$

with f smooth and nonnegative, we show that the uniqueness holds at different ranges of q depending whether λ is positive or not. Indeed, when $\lambda > 0$, uniqueness can be proved for $q \geq 1$, while it only holds for $q \geq p - 1$ in case $\lambda = 0$. In this latter case, however, the inconvenience due to the lack of zero order terms can be overcome in two particular situations; one is when the right-hand side $f(x)$ has a strict sign, another one is when either u or v is a strict sub or super solution.

Let us also refer the reader to [5] for a totally different approach, based on viscosity solutions, to the problem of comparison principle when the coercivity of zero order terms is missing. However, the use of viscosity solutions would not be suitable in our context since we deal with only measurable dependence with respect to x.

We present now our results under general structure conditions for the equation

$$-\Delta_p u + \mathcal{H}(x, u, Du) = 0 \qquad \text{in } \Omega\,, \tag{1.3}$$

where $\mathcal{H}(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, measurable with respect to x and C^1 with respect to (s, ξ).

We first consider the case in which $\mathcal{H}(x, s, \xi)$ satisfies a convexity-type condition with respect to ξ (suitably rescaled for the $(p-1)$–growth) as in the model problem (1.2). Notice that this condition will be required either to hold for every ξ, when there is no coercivity from the zero order term in the equation, or to hold only at infinity if $\mathcal{H}_s > 0$. A first simple result in this direction, though not the most general of our paper, is the following.

Theorem 1.1. *Let u, v be $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ sub and supersolutions to (1.3) respectively, and assume that $\mathcal{H}(x,s,\xi)$ satisfies one of the two following set of assumptions for either $\tau = 1$ or $\tau = -1$:*

i) *there exists $c > 0$ and $\lambda > 0$ such that*

$$\begin{aligned} &\tau\left[\mathcal{H}_\xi(x,s,\xi)\cdot\xi - (p-1)\mathcal{H}(x,s,\xi)\right] \geq -c\,,\\ &|\mathcal{H}_\xi(x,s,\xi)| \leq c(1+|\xi|^{p-1})\,,\\ &\mathcal{H}_s(x,s,\xi) \geq \lambda\,, \qquad \lambda > 0 \end{aligned} \tag{1.4}$$

for a.e. $x \in \Omega$, for any $\xi \in \mathbb{R}^N$ and for $s \in [-M, M]$, $M = \max(\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)})$;

ii) *there exists $c > 0$ such that*

$$\begin{aligned} &\tau\left[\mathcal{H}_\xi(x,s,\xi)\cdot\xi - (p-1)\mathcal{H}(x,s,\xi)\right] \geq 0\\ &|\mathcal{H}_\xi(x,s,\xi)| \leq c|\xi|^{p-1}\\ &\mathcal{H}_s(x,s,\xi) \geq 0\,, \end{aligned} \tag{1.5}$$

for a.e. $x \in \Omega$, for any $\xi \in \mathbb{R}^N$ and for $s \in [-M, M]$, $M = \max(\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)})$.

If $u \leq v$ at $\partial\Omega$, then $u \leq v$ in Ω.

Let us first recall that in order to have bounded solutions it is enough for instance that $f \in L^r(\Omega)$ with $r > \frac{N}{p}$.

Observe that the result of Theorem 1.1 applies in particular to equation (1.2) with $\lambda > 0$ and $p - 1 \leq q \leq p$, while the case $\lambda = 0$ is included in assumption (1.5) only when $q = p$ and $f \geq 0$.

The second result that we present deals with the case that the Hamiltonian has natural growth (without any convexity-type requirement) and some extra coercivity comes from either $\mathcal{H}_s(x,s,\xi)$ or $\mathcal{H}(x,s,0)$. In the first case, the result extends the previous one by weakening the hypothesis (1.4). In the second case, the growth of $\mathcal{H}$ is more general than in (1.5), but a strict condition on $\mathcal{H}(x,s,0)$ is also required. The case in which one between the super and the subsolution is strict is also considered here.

Theorem 1.2. *Let u, v be $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ sub and supersolutions to (1.3) respectively, and assume that $\mathcal{H}(x,s,\xi)$ satisfies one of the following set of assumptions for either $\tau = 1$ or $\tau = -1$:*

i) *there exist $c > 0$ and $\lambda > 0$ such that*

$$\begin{aligned} &\tau[\mathcal{H}_\xi(x,s,\xi)\cdot\xi - (p-1)\mathcal{H}(x,s,\xi)] \geq -c(1+|\xi|^p)\,,\\ &|\mathcal{H}_\xi(x,s,\xi)| \leq c(1+|\xi|^{p-1})\,,\\ &\mathcal{H}_s(x,s,\xi) \geq \lambda\,, \qquad \lambda > 0\,, \end{aligned} \tag{1.6}$$

for a.e. $x \in \Omega$, for any $\xi \in \mathbb{R}^N$ and for $s \in [-M, M]$, $M = \max(\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)})$;

ii) *there exists $c > 0$ such that*

$$\begin{aligned} &\tau[\mathcal{H}_\xi(x,s,\xi)\cdot\xi - (p-1)\mathcal{H}(x,s,\xi)] \geq -c|\xi|^p + \mu \quad \textit{for some } \mu > 0,\\ &|\mathcal{H}_\xi(x,s,\xi)| \leq c(1+|\xi|^{p-1})\,,\\ &\mathcal{H}_s(x,s,\xi) \geq 0\,, \end{aligned} \tag{1.7}$$

for a.e. $x \in \Omega$, for any $\xi \in \mathbb{R}^N$ and for $s \in [-M, M]$, $M = \max(\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)})$;

iii) *hypothesis (1.6) holds with* $\lambda = 0$ *and at least one between the subsolution* u *or the supersolution* v *is strict.*

If $u \leq v$ *at* $\partial\Omega$*, then* $u \leq v$ *in* Ω*.*

Assumption (1.6) is satisfied in particular when H has natural growth, namely when $|\mathcal{H}(x,s,\xi)| \leq c(1+|\xi|^p)$ and $|\mathcal{H}_\xi(x,s,\xi)| \leq c(1+|\xi|^{p-1})$. However $\mathcal{H}(x,s,0)$ does not necessarily need to be bounded from the conditions in (1.6). Notice that this result applies to problem (1.2) when $\lambda > 0$ and $q \in [1,p]$, extending the range of q which were covered by Theorem 1.1 .

On another hand, (1.7) implies a strict sign on $\mathcal{H}(x,s,0)$; in particular this case applies to the model problem (1.2) when $\lambda \geq 0$, $q \in [1,p]$ and $f(x)$ has a strict sign. It is remarkable that this condition extends the uniqueness range to the case $1 \leq q < p-1$. In some sense, the strict sign of f replaces the role of zero order terms and prevents the nonuniqueness phenomena as in ordinary differential equations.

A similar situation happens whenever we compare strict sub or supersolutions of (1.3), as expressed by the case (iii) in Theorem 1.2.

The proof of Theorems 1.1 and 1.2 follows the approach introduced by Barles and Murat in [6] which relies on a systematic use of changes of unknown. This tool, which we present in its general version in Theorem 2.2, is essentially restricted to bounded solutions, since the equation is invariant through changes of unknown only in the class of $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ solutions. Let us stress that, with respect to [6], we devote a special attention to handle the cases where the coercivity of the zero order term is missing.

We also present below a third result where, again, no coercivity is assumed on $\mathcal{H}_s$. Here we only use the convexity properties of $\mathcal{H}$ jointly with scaling arguments, so this approach is likely to be extended to a context of unbounded solutions. Let us mention that here $\mathcal{H}(x,s,\xi)$ can be assumed to be only continuous in (s,ξ) and measurable in x.

Theorem 1.3. *Let* u, v *be* $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *sub and supersolutions to (1.3) respectively, and assume that* $\mathcal{H}(x,s,\xi)$ *satisfies the following condition:*
for every $\varepsilon > 0$ *sufficiently small there exists* c_ε *such that*

$$\mathcal{H}(x,s,\xi) - (1-\varepsilon)^{p-1}\mathcal{H}(x, \frac{t+\varepsilon K}{1-\varepsilon}, \frac{\eta}{1-\varepsilon}) \leq c_\varepsilon |\xi-\eta|^{p-1}\,(1+|\xi-\eta|) \qquad \forall s \leq t\,, \tag{1.8}$$

for some $K > 0$*, for a.e.* $x \in \Omega$*, for any* $\xi, \eta \in \mathbb{R}^N$ *and for* $s, \frac{t+\varepsilon K}{1-\varepsilon} \in [-M, M]$*,* $M = \max(\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)})$*.*

If $u \leq v$ *at* $\partial\Omega$*, then* $u \leq v$ *in* Ω*.*

To compare with the previous statements, we observe that condition (1.8) implies that $\mathcal{H}(x,s,\xi)$ is nondecreasing in s; on the other hand if $\mathcal{H}$ is independent of s, it implies that $\mathcal{H}(x,0) \leq 0$. This latter case applies in particular to the model problem (1.2) when $\lambda = 0$, $f \geq 0$ and $q > p-1$. More generally, Theorem 1.3 can be applied to the equation

$$-\Delta_p u + g(u)|Du|^q = f(x) \qquad \text{in } \Omega\,, \tag{1.9}$$

when g is nondecreasing, $f \in L^1(\Omega)$, $f \geq 0$ and $q \in [p-1,p]$. The limiting case $q = p-1$ is quite delicate here and not necessarily included in all examples, but it is admitted if, for instance, g is increasing. The reader is referred to Corollary 3.1.

We stress that a version of Theorem 1.3 specifically devoted to the case that $\lambda > 0$ or $f(x) > 0$ in equation (1.2) is also proved in Section 3 (see Theorem 3.2 and Corollary 3.3).

Finally, we conclude our paper by proving a Strong Comparison Principle for equation (1.1). To this purpose, we exploit the approach which goes through the linearization of the p-Laplace operator, following the ideas in [10]. As mentioned above, this method requires some non trivial preliminary estimates on $|Du|^{p-2}$ in order to have a weighted Poincaré-Sobolev inequality available for the operator linearized along a solution. To fix the ideas, we deal with locally Lipschitz solutions of

$$-\Delta_p u + H(x, Du) = f(x) \qquad \text{in } \Omega . \tag{1.10}$$

with H a smooth (say C^1) function and f smooth enough.

The estimates needed in order to prove weighted Poincaré-Sobolev inequality require, at least with the usual method, that

$$f \in W^{1,\infty}_{loc}(\Omega) , \qquad f(x) > 0 \quad \text{in } \ \Omega , \tag{1.11}$$

and moreover that H is C^1 with respect to ξ and it satisfies

$$H(x, 0) = 0 , \qquad |H_\xi(x, \xi)| \le \gamma\, |\xi|^{\frac{p-2}{2}} \qquad \forall \xi \ : |\xi| \le 1 \quad \text{ for a.e. } x \in \Omega . \tag{1.12}$$

Notice that no sign condition is assumed on H, so the case when $f(x) < 0$ can be recovered in the result below by simply changing u into $-u$ in the equation.

At the price of imposing much heavier conditions both on the data as well as on the solutions, we obtain a comparison principle in a strong form. Such a fact has significant applications, especially to the p-Laplace ergodic problem which is studied in [15]. This has been indeed our motivation to look for a comparison principle in a strong form. The main result in this framework is the following.

Theorem 1.4 (Strong Comparison Principle)**.** *Let $u, v \in W^{1,\infty}_{loc}(\Omega)$. Assume that at least one between u or v is a weak solution to (1.10), and let $f(x)$, $H(x, \xi)$ satisfy (1.11)–(1.12). Assume that*

$$-\Delta_p u + H(x, Du) \le -\Delta_p v + H(x, Dv) , \qquad u \le v \quad \text{ in } \ \Omega$$

and that Ω is connected. Then $u \equiv v$ in Ω, unless $u < v$ in Ω.

We remark that, after the completion of this work, we learned that a strong comparison principle in a similar setting has been proved independently in [17] motivated by different applications which concern the study of qualitative properties of solutions of p-Laplacian type equations.

To conclude, we briefly present the plan of the paper. In Section 3, we prove Theorem 1.1 and 1.2, while Section 2 is devoted to Theorem 1.3 and some related results. Overall, Theorems 1.1–1.3 are the main weak comparison principles that we prove in this paper, although some more variants or particular cases will be specifically addressed. The strong comparison principle is left to Section 4.

Finally we collect in the Appendix some further minor results, which we could not find in any previous reference and might be interesting as well. They are concerned with the case of equation (1.1) with positive zero order term (i.e. $\lambda > 0$) and either Lipschitz Hamiltonian, in which case the comparison holds even for merely $W^{1,p}$ solutions, or Lipschitz solutions.

Notations and basic inequalities. In all of this paper, we assume that $p \ge 2$.

Here we recall some classical inequalities such as

$$c_p\, (|a|^{p-2} + |b|^{p-2})|a - b|^2 \le (|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b) \qquad \forall a, b \in \mathbb{R}^N , \tag{1.13}$$

which implies, since $p \ge 2$,

$$c_p\, |a - b|^p \le (|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b) \qquad \forall a, b \in \mathbb{R}^N . \tag{1.14}$$

Let us recall the meaning that we give to solutions that are considered in this paper. As usual when the p-Laplacian is involved, solutions are understood in a weak sense, since they are not smooth enough to have "classical" sense. Specifically, we say that $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a subsolution (supersolution, respectively) of

$$-\Delta_p u + \mathcal{H}(x,u,Du) = 0 \qquad \text{in } \Omega\,,$$

if $\mathcal{H}(x,u,Du) \in L^1(\Omega)$ and

$$\int_\Omega |Du|^{p-2} Du D\phi + \int_\Omega \mathcal{H}(x,u,Du)\phi \le\ (\ge)\, 0\,, \qquad \forall \phi \in C_c^1(\Omega)\,, \quad \phi \ge 0\,.$$

Moreover, by density, it is sufficient to choose the test functions in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ in order to give sense to all the terms in the integrals above.

Of course, a solution is both a subsolution and a supersolution.

Whenever we deal with a subsolution and a supersolution $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, M indicates the following constant $M = \max(\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)})$. Moreover, when we say "$u \le v$ on $\partial\Omega$" we mean that $(u-v)^+ \in W_0^{1,p}(\Omega)$, if $u, v \in W^{1,p}(\Omega)$.

Finally, by c we denote possibly different constants, whose value can change from line to line. If needed, we specify the dependence of such a constant by a suitable parameter as an index.

2. Comparison via linearization of the lower order term and change of unknown

In this section we consider the general equation

$$-\,\Delta_p u + \mathcal{H}(x,u,Du) = 0 \qquad \text{in } \Omega\,, \tag{2.1}$$

where, as before, $p > 2$ and $H(x,s,\xi)$ is measurable with respect to x and C^1 with respect to (s,ξ). We are going to exploit general structure conditions over $\mathcal{H}$ under which the comparison principle can be proved to hold. This is in the spirit of the results proved by Barles and Murat for the case $p = 2$ (see [6] and [4]). Let us point out that the possibility to extend their results to the case $p > 2$ was already mentioned and suggested by Barles and Murat at the end of their work, although a systematic investigation of this case had not been pursued so far.

The peculiarity of the case $p > 2$ is clear in the following basic starting point concerned with the linearization of the Hamiltonian (to be compared with the proof of Proposition A.3 in the Appendix).

Proposition 2.1. *Let u, v be $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ sub and supersolutions to (2.1) respectively, and let $\mathcal{H}(x,s,\xi)$ satisfy the following condition*

$$\exists \delta > 0: \qquad \mathcal{H}_s(x,s,\xi) - \delta |\mathcal{H}_\xi(x,s,\xi)|^{p'} \ge 0\,, \qquad \forall s \in [-M,M]\,,\ \forall \xi \in \mathbb{R}^N \text{ and a.e. } x \in \Omega. \tag{2.2}$$

where $M = \max(\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)})$. If $u \le v$ at $\partial\Omega$, then $u \le v$ in Ω.

Moreover, the conclusion holds for u, v only belonging to $W^{1,p}(\Omega)$ provided (2.2) holds for all $s \in \mathbb{R}$.

Proof. Using the inequalities solved by u, v we deduce that

$$-\Delta_p u + \Delta_p v + \mathcal{H}(x,u,Du) - \mathcal{H}(x,v,Dv) \le 0 \qquad \text{in } \Omega\,.$$

Let us denote by $w = u - v$ and let $S(\sigma)$ be a $C^1[0,+\infty)$ increasing function such that $S(0) = 0$, so that we are allowed to take $S(w^+)$ as test function in the above inequality. Thus we have:

$$\int_\Omega |Dw|^p S'(w^+) + \int_\Omega [\mathcal{H}(x,u,Du) - \mathcal{H}(x,v,Dv)] S(w^+) \le 0\,. \tag{2.3}$$

We use now that

$$\mathcal{H}(x,u,Du)-\mathcal{H}(x,v,Dv)=\int_0^1\frac{d}{dt}\mathcal{H}(x,z_t,Dz_t)dt=\int_0^1[\mathcal{H}_s(x,z_t,Dz_t)w+\mathcal{H}_\xi(x,z_t,Dz_t)Dw]dt\,,$$

where $z_t=tu+(1-t)v$. Consequently we have that

$$\int_\Omega|Dw|^pS'(w^+)+\int_\Omega\int_0^1[\mathcal{H}_s(x,z_t,Dz_t)w+\mathcal{H}_\xi(x,z_t,Dz_t)Dw]dtS(w^+)\le 0\,.$$

Since, by Young inequality, for any $\theta\in(0,1)$ we have

$$\int_0^1[\mathcal{H}_\xi(x,z_t,Dz_t)Dw]dtS(w^+)\le\theta|Dw|^pS'(w^+)+c_\theta\int_0^1|\mathcal{H}_\xi(x,z_t,Dz_t)|^{p'}dt\frac{S(w^+)^{p'}}{S'(w^+)^{\frac{1}{p-1}}}\,,$$

we get from (2.3)

$$(1-\theta)\int_\Omega|Dw|^pS'(w^+)+\int_\Omega\int_0^1\left[\mathcal{H}_s(x,z_t,Dz_t)w-c_\theta|\mathcal{H}_\xi(x,z_t,Dz_t)|^{p'}\left[\frac{S(w^+)}{S'(w^+)}\right]^{\frac{1}{p-1}}\right]S(w^+)\le 0$$

Here we choose, for any $\delta>0$, $S(s)=\psi_{\delta^{p-1}}(s)$, as defined in (A.4), that, according to (A.5), is such that $\frac{S(w)}{S'(w)}=w^{p-1}\delta^{p-1}$. Thus we deduce that

$$(1-\theta)\int_\Omega|Dw|^pS'(w^+)+\int_\Omega\int_0^1\left[\mathcal{H}_s(x,z_t,Dz_t)-\delta c_\theta|\mathcal{H}_\xi(x,z_t,Dz_t)|^{p'}\right]w^+S(w^+)\le 0. \tag{2.4}$$

Since (2.2) holds, choosing δ suitably small we conclude that $u\le v$ in Ω.

Finally, notice that $S(w)$ belongs to $L^\infty(\Omega)$ even if u,v are not; therefore, if (2.2) holds for all $s\in\mathbb{R}$ the proof remains valid for u,v only belonging to $W^{1,p}(\Omega)$. □

Henceforth, we borrow the strategy of Barles and Murat, consisting in a systematic search for a change of unknown which would reduce our equation into the conditions given by the above Proposition.

Before stating the general principle, let us fix some notations. We denote by φ a C^3 monotone function such that $\varphi'\neq 0$, in a way that $u=\varphi(z)$ is a possible change of unknown in the equation, provided $[-M,M]$ is contained in the range of φ. In this case, we set $K>0$ such that $\varphi^{-1}([-M,M])\subset[-K,K]$.

Theorem 2.2. *Let u,v be $W^{1,p}(\Omega)\cap L^\infty(\Omega)$ sub and supersolutions to (2.1) respectively. Assume that there exists $\delta>0$ and a function $\varphi(s)$ such that $[-M,M]$ is contained in the range of φ and*

$$\begin{aligned}&-(p-1)\left(\frac{\varphi''(s)}{\varphi'(s)}\right)'|\xi|^p+\frac{1}{|\varphi'(s)|^{p-2}}\mathcal{H}_s(x,\varphi(s),\xi\varphi'(s))\\&+\frac{\varphi''(s)}{|\varphi'(s)|^p}\left[\mathcal{H}_\xi(x,\varphi(s),\xi\varphi'(s))\cdot(\varphi'(s)\xi)-(p-1)\mathcal{H}(x,\varphi(s),\xi\varphi'(s))\right]\\&\ge\delta\left|-p(p-1)\frac{\varphi''(s)}{\varphi'(s)}|\xi|^{p-2}\xi+\frac{1}{|\varphi'(s)|^{p-2}}\mathcal{H}_\xi(x,\varphi(s),\xi\varphi'(s))\right|^{p'},\end{aligned} \tag{2.5}$$

for a.e. $x\in\Omega$, for any $\xi\in\mathbb{R}^N$ and for $s\in[-K,K]$. If $u\le v$ at $\partial\Omega$, then $u\le v$ in Ω.

Proof. Without loss of generality, we suppose that $\varphi'(s)>0$.

Observe that if u (v respectively) is a subsolution (supersolution) to (2.1), then $z=\varphi^{-1}(u)$ is a subsolution ($\varphi^{-1}(v)$ is a supersolution) to

$$-\Delta_pz+\widetilde{\mathcal{H}}(x,z,Dz)=0\qquad\text{in }\Omega\,,$$

where

$$\widetilde{\mathcal{H}}(x,z,Dz) = -(p-1)\frac{\varphi''(z)}{\varphi'(z)}|Dz|^p + \frac{1}{|\varphi'(z)|^{p-2}\varphi'(z)}\mathcal{H}(x,\varphi(z),Dz\varphi'(z))\,.$$

Here we just look for some φ such that the hypothesis (2.2) is fulfilled. Computing we have

$$\begin{aligned}\widetilde{\mathcal{H}}_s(x,s,\xi) &= -(p-1)\left(\frac{\varphi''(s)}{\varphi'(s)}\right)'|\xi|^p + \frac{1}{|\varphi'(s)|^{p-2}}\mathcal{H}_s(x,\varphi(s),\xi\varphi'(s))\\ &+\frac{\varphi''(s)}{|\varphi'(s)|^p}\Big[\varphi'(s)\xi\cdot\mathcal{H}_\xi(x,\varphi(s),\xi\varphi'(s)) - (p-1)\mathcal{H}(x,\varphi(s),\xi\varphi'(s))\Big]\,,\end{aligned}$$

while

$$\widetilde{\mathcal{H}}_\xi(x,s,\xi) = -p(p-1)\frac{\varphi''(s)}{\varphi'(s)}|\xi|^{p-2}\xi + \frac{1}{|\varphi'(s)|^{p-2}}\mathcal{H}_\xi(x,\varphi(s),\xi\varphi'(s))\,.$$

Therefore, if (2.5) is satisfied for some $\delta>0$, then (2.2) holds for $\widetilde{\mathcal{H}}$. By Proposition 2.1, we deduce $\varphi^{-1}(u)\le\varphi^{-1}(v)$, and so $u\le v$. □

Remark 2.3. We stress that the comparison principle of Theorem 2.2 may be proved to hold for merely $W^{1,p}(\Omega)$ sub/supersolutions if one can find a change of variable $\varphi(s)$ mapping $W^{1,p}(\Omega)$ into $W^{1,p}(\Omega)$ and such that (2.5) holds for any s. Indeed, the Proposition 2.1 can be applied in this case to $W^{1,p}(\Omega)$ sub/super solutions of the transformed equation.

The above result, despite its generality, is not so immediate to be applied. The main corollaries we obtain are Theorems 1.1 and 1.2 of the Introduction, which we prove now. To this purpose, observe that condition (2.5) is satisfied if we prove that, for some $\delta>0$, we have

$$\begin{aligned}&-(p-1)\left(\frac{\varphi''(s)}{\varphi'(s)}\right)'|\xi|^p + \frac{1}{|\varphi'(s)|^{p-2}}\mathcal{H}_s(x,\varphi(s),\xi\varphi'(s))\\ &+\frac{\varphi''(s)}{|\varphi'(s)|^p}\left[\mathcal{H}_\xi(x,\varphi(s),\xi\varphi'(s))\cdot(\varphi'(s)\xi) - (p-1)\mathcal{H}(x,\varphi(s),\xi\varphi'(s))\right]\\ &\ge\delta\left[\Big|\frac{\varphi''(s)}{\varphi'(s)}\Big|^{p'}|\xi|^p + \frac{1}{|\varphi'(s)|^{p'(p-2)}}|\mathcal{H}_\xi(x,\varphi(s),\xi\varphi'(s))|^{p'}\right],\end{aligned}\tag{2.6}$$

for a.e. $x\in\Omega$, for any $\xi\in\mathbb{R}^N$ and for $s\in[-K,K]$.

Proof. [of Theorem 1.1] First, let us set the following function:

$$\varphi(s) = \gamma\int_0^s e^{-r^2}dr + M \qquad \text{with } \gamma, M>0\,,$$

where γ is taken sufficiently large so that the range of φ contains $[-M,M]$.

Case i). Assume that (1.4) holds with $\tau=1$. In this case (2.6) is proved if we show that $\varphi''\ge 0$ and, for some $\delta>0$,

$$-(p-1)\left(\frac{\varphi''(s)}{\varphi'(s)}\right)'|\xi|^p + \frac{\lambda}{|\varphi'(s)|^{p-2}} - c\frac{\varphi''(s)}{|\varphi'(s)|^p} \ge \delta\left[\Big|\frac{\varphi''(s)}{\varphi'(s)}\Big|^{p'} + c\,|\varphi'(s)|^{p'}\right]|\xi|^p + \delta\frac{c}{|\varphi'(s)|^{p'(p-2)}}\,.\tag{2.7}$$

Notice that $s\le 0$ whenever $\varphi(s)\le M$, so we have $\varphi''(s)\ge 0$. Moreover we have

$$\begin{aligned}&-(p-1)\left(\frac{\varphi''(s)}{\varphi'(s)}\right)'|\xi|^p + \frac{\lambda}{|\varphi'(s)|^{p-2}} - c\frac{\varphi''(s)}{|\varphi'(s)|^p}\\ &= 2(p-1)|\xi|^p + \gamma^{-(p-2)}\,e^{(p-2)s^2}\left[\lambda - c\frac{2|s|e^{s^2}}{\gamma}\right].\end{aligned}$$

Recall that s lies in a compact set, which can be chosen uniformly bounded with respect to γ, whenever this is large. Therefore, if we first choose γ large enough so that $\lambda - c\frac{|s|e^{s^2}}{\gamma} > 0$, then inequality (2.7) will be satisfied for a choice of δ sufficiently small.

Case ii). Assume that (1.5) holds, still with $\tau = 1$. In this case (2.6) is satisfied provided we have $\varphi'' \geq 0$ and, for some $\delta > 0$,

$$- (p-1)\left(\frac{\varphi''(s)}{\varphi'(s)}\right)'|\xi|^p \geq \delta\left[\left|\frac{\varphi''(s)}{\varphi'(s)}\right|^{p'}|\xi|^p + c\,|\varphi'(s)|^{p'}|\xi|^p\right]. \tag{2.8}$$

We make the same choice of φ as before. Now (2.8) reduces to

$$2(p-1) \geq \delta\left[\left|\frac{\varphi''(s)}{\varphi'(s)}\right|^{p'} + c\,|\varphi'(s)|^{p'}\right] = \delta\left[\left|2s\right|^{p'} + c\,\gamma^{p'}\,e^{-p's^2}\right].$$

Since s belongs to a compact set, the inequality is satisfied up to choosing δ sufficiently small.

Finally, if (1.4) or (1.5) hold with $\tau = -1$, then $\widetilde{\mathcal{H}}(x, s, \xi) = -\mathcal{H}(x, -s, -\xi)$ satisfies the same assumptions with $\tau = 1$, so we reduce to the previous case by changing u into $-u$ and v into $-v$. □

Next, with suitable choices of the change of unknown, we deal with Theorem 1.2.

Proof. [of Theorem 1.2] Here we choose

$$\varphi(s) = -\beta\,(p-1)\log(\alpha + e^{-\frac{k}{p-1}s}) \qquad \text{with} \qquad \alpha, k > 0 \text{ and } \beta \in \mathbb{R}\,. \tag{2.9}$$

In the sequel we choose α sufficiently small, compared to $|\beta|$, so that the range of φ contains $[-M, M]$ (i.e. we take $\alpha < e^{-\frac{M}{|\beta|(p-1)}}$).

Case i). Assume that (1.6) holds with $\tau = 1$. We choose $\beta < 0$, so that $\varphi'' \geq 0$ (conversely, we take $\beta > 0$ if it holds with $\tau = -1$). Then, we see that (2.6) is satisfied if we show that

$$\begin{aligned} &\left\{-(p-1)\left(\frac{\varphi''(s)}{\varphi'(s)}\right)' - c|\varphi''(s)|\right\}|\xi|^p + \frac{1}{|\varphi'(s)|^p}\left\{\lambda\varphi'(s)^2 - c|\varphi''(s)|\right\} \\ &\geq \delta\left[\left|\frac{\varphi''(s)}{\varphi'(s)}\right|^{p'} + c\,|\varphi'(s)|^{p'}\right]|\xi|^p + \delta\,\frac{c}{|\varphi'(s)|^{p'(p-2)}} \qquad \forall s \in [-K, K]\,, \quad \forall \xi \in \mathbb{R}^N\,. \end{aligned} \tag{2.10}$$

Observe that

$$-(p-1)\left(\frac{\varphi''(s)}{\varphi'(s)}\right)' - c|\varphi''(s)| = \frac{\alpha k^2}{p-1}(1 - c\,|\beta|)\frac{e^{-\frac{k}{p-1}s}}{(\alpha + e^{-\frac{k}{p-1}s})^2}$$

and thus we fix $|\beta| < \frac{1}{c}$ so that the right hand side above is positive. We also notice that

$$\lambda\varphi'(s)^2 - c|\varphi''| = \frac{|\beta|\,k^2e^{-\frac{k}{p-1}s}}{(\alpha + e^{-\frac{k}{p-1}s})^2}\left(\lambda\,|\beta|\,e^{-\frac{k}{p-1}s} - \frac{c\,\alpha}{p-1}\right)$$

and since $e^{-\frac{k}{p-1}s} = e^{-\frac{\varphi(s)}{\beta(p-1)}} - \alpha$ this yields

$$\lambda\varphi'(s)^2 - c|\varphi''| = \frac{|\beta|\,k^2e^{-\frac{k}{p-1}s}}{(\alpha + e^{-\frac{k}{p-1}s})^2}\left[\lambda\,|\beta|\,e^{-\frac{\varphi(s)}{\beta(p-1)}} - \alpha\left(\lambda\,|\beta| + \frac{c}{p-1}\right)\right].$$

Since $\varphi(s) \in [-M, M]$, we can choose α sufficiently small so that this term is positive, too. Therefore, both terms in the left-hand side of (2.10) can be made positive and the inequality is satisfied by choosing δ sufficiently small.

Case ii). With the same choice as before, we have $\varphi''(s) = \tau|\varphi''(s)|$ and we verify in a similar way that

$$\left\{-(p-1)\left(\frac{\varphi''(s)}{\varphi'(s)}\right)' - c|\varphi''(s)|\right\}|\xi|^p + \mu\frac{\varphi''(s)}{|\varphi'(s)|^p}$$
$$\geq \delta\left[\left|\frac{\varphi''(s)}{\varphi'(s)}\right|^{p'} + c\,|\varphi'(s)|^{p'}\right]|\xi|^p + \delta\,\frac{c}{|\varphi'(s)|^{p'(p-2)}} \qquad \forall s \in [-K, K]\,, \quad \forall \xi \in \mathbb{R}^N\,.$$

Therefore (2.6) is satisfied and we conclude again by Theorem 2.2.

Case iii). Suppose without loosing generality that u is a strict subsolution. This means that, for some $\rho > 0$, we have

$$-\Delta_p u + \mathcal{H}(x, u, Du) + \rho \leq 0 \qquad \text{in } \Omega\,.$$

With the same notations as in Theorem 2.2, by setting $u = \varphi(\tilde{u})$ and $v = \varphi(\tilde{v})$ we have now

$$-\Delta_p\tilde{u} + \Delta_p\tilde{v} + \widetilde{\mathcal{H}}(x, \tilde{u}, D\tilde{u}) - \widetilde{\mathcal{H}}(x, \tilde{v}, D\tilde{v}) + \frac{\rho}{\varphi'(\tilde{u})^{p-1}} \leq 0 \qquad \text{in } \Omega\,.$$

Hence, multiplying by the test function, $\forall\delta > 0$, $S(w^+) = \psi_{\delta^{p-1}}(w^+)$ as in (A.4), and proceeding as in Theorem 2.2 we obtain, for some $\theta \in (0, 1)$,

$$(1-\theta)\int_\Omega |Dw|^p S'(w^+) +$$
$$\int_\Omega\left[\int_0^1\left(\widetilde{\mathcal{H}}_s(x, z_t, Dz_t) - \delta c_\theta|\widetilde{\mathcal{H}}_\xi(x, z_t, Dz_t)|^{p'}\right)dt \;+\frac{\rho}{|\varphi'(u)|^{p-1}\;w^+}\right]w^+S(w^+) \leq 0\,.$$

Thanks to the choice of φ, we have $|\varphi'| \leq |\beta|\,k$, and, since $w \leq 2K$, we get

$$(1-\theta)\int_\Omega |Dw|^p S'(w^+) +$$
$$\int_\Omega\left[\int_0^1\left(\widetilde{\mathcal{H}}_s(x, z_t, Dz_t) - \delta c_\theta|\widetilde{\mathcal{H}}_\xi(x, z_t, Dz_t)|^{p'}\right)dt \;+\frac{\rho}{(\beta\,k)^{p-1}2K}\right]w^+S(w^+) \leq 0\,.$$

Now, on account of (2.9), we conclude if we can find δ such that,

$$\left\{-(p-1)\left(\frac{\varphi''(s)}{\varphi'(s)}\right)' - c|\varphi''(s)|\right\}|\xi|^p - c\frac{|\varphi''(s)|}{|\varphi'(s)|^p} + \frac{\rho}{(\beta\,k)^{p-1}2K}$$
$$\geq \delta\left[\left|\frac{\varphi''(s)}{\varphi'(s)}\right|^{p'} + c|\varphi'(s)|^{p'}\right]|\xi|^p + \delta\,\frac{c}{|\varphi'(s)|^{p'(p-2)}} \qquad \forall s \in [-K, K]\,, \quad \forall \xi \in \mathbb{R}^N\,. \tag{2.11}$$

But recall that $e^{-\frac{k}{p-1}s} = e^{-\frac{\varphi(s)}{\beta(p-1)}} - \alpha$, hence $|\varphi'| \geq \beta\,k c_{\alpha,\beta,M}$ for some constant $c_{\alpha,\beta,M}$ only depending on α, β, M and not on k. Since $|\frac{\varphi''(s)}{\varphi'(s)}| \leq \frac{k}{p-1}$, we deduce that

$$\frac{|\varphi''(s)|}{|\varphi'(s)|^p} \leq \frac{k}{(\beta\,k)^{p-1}c_{\alpha,\beta,M}^{p-1}(p-1)}$$

so that

$$-c\frac{|\varphi''(s)|}{|\varphi'(s)|^p} + \frac{\rho}{(\beta\,k)^{p-1}2K} \geq \frac{1}{(\beta\,k)^{p-1}}\left[\frac{\rho}{2K} - \frac{c\,k}{c_{\alpha,\beta,M}^{p-1}(p-1)}\right].$$

Once α and β are fixed, as in the previous theorem, we choose now k small enough so that this latter term is positive. Then, (2.11) holds for a sufficiently small $\delta > 0$. □

Remark 2.4. As already explained in [6], the C^1 character of $\mathcal{H}$ could be relaxed by only requiring that $\mathcal{H}(x, s, \xi)$ is locally Lipschitz with respect to (s, ξ). This requires an approximation argument and we refer the reader to [6, Section 3] where details are given.

3. Comparison via scaling and convexity argument

The aim of this Section is to prove Theorem 1.3 and some extensions, mainly exploiting a sort of convexity of the Hamiltonian term.

Let us stress that assumption (1.8) implies that $\mathcal{H}(x,s,\xi)$ is nondecreasing with respect to s (as readily seen by taking $\xi=\eta$ and letting $\varepsilon\to 0$); in particular, there is no loss of generality by assuming that (1.8) holds for a constant K sufficiently large.

Proof. [of Theorem 1.3] We define $u_\varepsilon=(1-\varepsilon)u-\varepsilon K$ with $\varepsilon\in(0,1)$ and $K\ge \|u^-\|_{L^\infty(\Omega)}$ (so that $u_\varepsilon\le u$). Multiplying the inequality satisfied by u by $(1-\varepsilon)^{p-1}$ and subtracting the equation of v we get

$$-\Delta_p u_\varepsilon+\Delta_p v\le \mathcal{H}(x,v,Dv)-(1-\varepsilon)^{p-1}\mathcal{H}(x,\tfrac{u_\varepsilon+\varepsilon K}{1-\varepsilon},\tfrac{Du_\varepsilon}{1-\varepsilon})\,. \tag{3.1}$$

We set $m_\varepsilon=\ \operatorname{ess\,sup}_\Omega(u_\varepsilon-v)$ and we suppose, by contradiction, that $m_\varepsilon>0$. We choose $w=(u_\varepsilon-v-k)^+$ as test function in the above inequality, with $k\in(0,m_\varepsilon)$ and we use assumption (1.8) to estimate the right-hand side, obtaining

$$\begin{aligned}&\int_\Omega\big(|Du_\varepsilon|^{p-2}Du_\varepsilon-|Dv|^{p-2}Dv\big)D(u_\varepsilon-v-k)^+\\ &\le c_\varepsilon\int_\Omega|D(u_\varepsilon-v)|^{p-1}(u_\varepsilon-v-k)^+ + c_\varepsilon\int_\Omega|D(u_\varepsilon-v)|^{p}(u_\varepsilon-v-k)^+\,.\end{aligned}$$

Using (1.14) and Young inequality we get

$$\frac12\int_\Omega|Dw|^p\le c_\varepsilon\int_{A_k}w^p+c_\varepsilon\int_\Omega|Dw|^p w\,,$$

where $A_k=\{x\in\Omega\,:\,u_\varepsilon-v\ge k\}\cap\{x\in\Omega\,:\,Du_\varepsilon\ne Dv\}$ and we still denote by c_ε possibly different constants depending on ε. Since $w\le m_\varepsilon-k$, by choosing k sufficiently close to m_ε last term can be absorbed in the left-hand side and we deduce

$$\frac14\int_\Omega|Dw|^p\le c_\varepsilon\int_{A_k}w^p\,,$$

which implies, by Poincaré-Sobolev inequality, (here by p^* we denote the Sobolev conjugate of p, if $p<N$, or any number greater than p, if $p\ge N$)

$$\|w\|^p_{L^{p^*}(\Omega)}\le c_\varepsilon\int_{A_k}w^p\le c_\varepsilon\|w\|^p_{L^{p^*}(\Omega)}|A_k|^{1-\frac{p}{p^*}}\,.$$

Since $|A_k|\to 0$ as $k\to m_\varepsilon$, we conclude that $(u_\varepsilon-v-k)^+=0$ for some $k<m_\varepsilon$, getting a contradiction with the definition of m_ε. □

The easiest example for which hypothesis (1.8) holds is whenever $\mathcal{H}(x,s,\xi)=g(s)|\xi|^q-f(x)$, for some $q\ge p-1$ and g nondecreasing. We observe here that the growth $q=p-1$ is more delicate because of the homogeneous scaling, but the increasing character of g may give a contribution for this limiting case.

Corollary 3.1. *Let u,v be a $W^{1,p}(\Omega)\cap L^\infty(\Omega)$ sub and a supersolution, respectively, to*

$$-\Delta_p u+g(u)|Du|^q=f(x)\qquad in\ \ \Omega\,,$$

where g is a continuous nondecreasing function, $q\in(p-1,p]$ and $f\in L^1(\Omega)$, $f\ge 0$; assume also that $g(v)\ge 0$. If $u\le v$ at $\partial\Omega$, then $u\le v$ in Ω.

Moreover, the same result is true if $q=p-1$ and g is increasing.

Proof. First observe that there is no loss of generality in assuming that $g(u) \geq 0$. Indeed, let $s_0 := \inf\{s \in \mathbb{R} : g(s) = 0\}$; if $s_0 > -\infty$, then s_0 is a sub solution since $f \geq 0$ and so $\tilde{u} := \max(u, s_0)$ is still a sub solution. Since $g(v) \geq 0$ implies $v \geq s_0$, it would still hold that $\tilde{u} \leq v$ at $\partial\Omega$. Therefore, we could replace u with $\tilde{u}$ for which $g(\tilde{u}) \geq 0$ and, by proving $v \geq \tilde{u}$, we still deduce $v \geq u$. Thus, in the following we can and will assume that $g(u) \geq 0$ in Ω.

Let us take $K > M$, where we recall that we denote $M = \max(\|u\|_\infty, \|v\|_\infty)$. Then, for any $s \leq t$ we have $t \geq -M$, and so $t < \frac{t+\varepsilon K}{1-\varepsilon} \in [-M, M]$. Since g is nondecreasing, this means

$$g(s)|\xi|^q - g\big(\frac{t+\varepsilon K}{1-\varepsilon}\big)(1-\varepsilon)^{p-1}\frac{|\eta|^q}{(1-\varepsilon)^q} \leq g\big(\frac{t+\varepsilon K}{1-\varepsilon}\big)\left[|\xi|^q - \frac{|\eta|^q}{(1-\varepsilon)^{q-(p-1)}}\right].$$

Hence, when $q > p-1$ (1.8) holds because thanks to the convexity of $|Du|^q$ and since $g(u) \geq 0$.

If $q = p-1$, we assume that g is increasing. This implies that, for every $\varepsilon > 0$, there exists $\omega_\varepsilon > 0$ such that

$$g(s) \leq \frac{1}{1+\omega_\varepsilon} g\left(\frac{t+\varepsilon K}{1-\varepsilon}\right) \qquad \forall s \leq t\,. \tag{3.2}$$

Indeed, for any $s \leq t$ and $\frac{t+\varepsilon K}{1-\varepsilon} \in [-M, M]$ we have

$$g(s) \leq g(t) < g\left(\frac{t+\varepsilon K}{1-\varepsilon}\right)$$

because $-M \leq t < \frac{t+\varepsilon K}{1-\varepsilon}$ (due to $K > M$) and g is increasing. Since t lies in a compact set and g is continuous, there exists $\omega_\varepsilon > 0$ such that for every t

$$g(t) \leq \frac{1}{1+\omega_\varepsilon} g\left(\frac{t+\varepsilon K}{1-\varepsilon}\right)$$

and so (3.2) holds true. Therefore, we have

$$\begin{aligned}\mathcal{H}(x, s, \xi) - (1-\varepsilon)^{p-1}\mathcal{H}(x, \frac{t+\varepsilon M}{1-\varepsilon}, \frac{\eta}{1-\varepsilon}) &= g(s)|\xi|^{p-1} - g\big(\frac{t+\varepsilon K}{1-\varepsilon}\big)|\eta|^{p-1} \\ &\leq g\big(\frac{t+\varepsilon K}{1-\varepsilon}\big)\left[\frac{1}{1+\omega_\varepsilon}|\xi|^{p-1} - |\eta|^{p-1}\right].\end{aligned}$$

By convexity (recall that $p > 2$), we have

$$\forall \delta \in (0,1) \qquad |\xi|^{p-1} \leq (1-\delta)\frac{|\eta|^{p-1}}{(1-\delta)^{p-1}} + \delta\frac{|\xi-\eta|^{p-1}}{\delta^{p-1}}\,.$$

Thus, choosing $\delta > 0$ such that

$$(1-\delta)^{p-2} = \frac{1}{1+\omega_\varepsilon}$$

and using that $g(u) \geq 0$, we deduce that

$$g\big(\frac{t+\varepsilon K}{1-\varepsilon}\big)\left[\frac{1}{1+\omega_\varepsilon}|\xi|^{p-1} - |\eta|^{p-1}\right] \leq c_\varepsilon|\xi-\eta|^{p-1}\,.$$

This concludes the verification of assumption (1.8). □

The next result goes in the same direction of Theorem 1.3, considering the cases in which $H(x, s, 0)$ has not a sign or if the subsolution (or the supersolution) is strict.

Theorem 3.2. *Let u, v be $W^{1,p}(\Omega)\cap L^\infty(\Omega)$ sub and supersolutions to (1.3) respectively, and assume that $\mathcal{H}(x,s,\xi)$ satisfies the following condition:*
for every $\varepsilon>0$ sufficiently small there exists c_ε such that

$$\begin{aligned}&\mathcal{H}(x,s,\xi)-(1-\varepsilon)^{p-1}\mathcal{H}(x,\tfrac{t+\varepsilon K}{1-\varepsilon},\tfrac{\eta}{1-\varepsilon})\\ &\le c_\varepsilon\big[|\xi-\eta|^p+\big(\max\big\{1,|\xi|^{\frac{p-2}{2}}+|\eta|^{\frac{p-2}{2}}\big\}\big)|\xi-\eta|\big]-\lambda(t-s)-\alpha_\varepsilon\end{aligned}\tag{3.3}$$

for a.e. $x\in\Omega$, for any $\xi,\eta\in\mathbb{R}^N$ and for s, $\frac{t+\varepsilon K}{1-\varepsilon}\in[-M,M]$, $M=\max(\|u\|_{L^\infty(\Omega)},\|v\|_{L^\infty(\Omega)})$ $s\le t$, where $\lambda,\alpha_\varepsilon\ge0$. and $\lambda+\alpha_\varepsilon>0$.

If $u\le v$ at $\partial\Omega$, then $u\le v$ in Ω.

Proof. We first construct a subsolution $u_\varepsilon=(1-\varepsilon)u-\varepsilon K$ with $\varepsilon\in(0,1)$ and $K>0$ to be fixed, $K\ge\|u^-\|_{L^\infty(\Omega)}$. Thus subtracting the inequality satisfied by v to the one by u_ε we deduce:

$$-\Delta_p u_\varepsilon+\Delta_p v\le\mathcal{H}(x,v,Dv)-(1-\varepsilon)^{p-1}\mathcal{H}\big(x,\tfrac{u_\varepsilon+\varepsilon K}{1-\varepsilon},\tfrac{Du_\varepsilon}{1-\varepsilon}\big)\,.\tag{3.4}$$

and (3.3) yields to

$$\lambda(u_\varepsilon-v)+\alpha_\varepsilon\ -\Delta_p u_\varepsilon+\Delta_p v\le c_\varepsilon\big[|Dv-Du_\varepsilon|^p+\big(\max\big\{1,|Dv|^{\frac{p-2}{2}}+|Du_\varepsilon|^{\frac{p-2}{2}}\big\}\big)|Dv-Du_\varepsilon|\big].$$

We set $m_\varepsilon=$ ess $\sup_\Omega(u_\varepsilon-v)$ and we suppose $m_\varepsilon>0$; we choose $(u_\varepsilon-v-k)^+=w$ as test function in the above inequality, with $k\in(0,m_\varepsilon)$ to be fixed.

Consequently we have, denoting $\mu=\lambda k$ if $\lambda>0$ or $\mu=\alpha_\varepsilon$ otherwise,

$$\begin{aligned}&\int_{A_k}\lambda\,w^2+\int_{A_k}\mu w+\frac12\int_{A_k}|Dw|^p+\frac12\int_{A_k}|Dw|^2(|Du_\varepsilon|^{p-2}+|Dv|^{p-2})\\ &\quad\le c_\varepsilon\int_{A_k}|Dw|^pw+c_\varepsilon\int_{A_k}\Big(1+|Du_\varepsilon|^{\frac{p-2}{2}}+|Dv|^{\frac{p-2}{2}}\Big)|Dw|w\,,\end{aligned}\tag{3.5}$$

where $A_k=\{x\in\Omega\,:\,u_\varepsilon-v\ge k\}\cap\{x\in\Omega\,:\,Du_\varepsilon\ne Dv\}$. Since, by Young inequality,

$$\begin{aligned}&c_\varepsilon\int_{A_k}(1+|Du_\varepsilon|^{\frac{p-2}{2}}+|Dv|^{\frac{p-2}{2}})|Dw|w\\ &\le\frac14\int_{A_k}(|Du_\varepsilon|^{p-2}+|Dv|^{p-2})|Dw|^2+\frac14\int_{A_k}|Dw|^p+c_\varepsilon\int_{A_k}w^2+w^{p'},\end{aligned}$$

joining it with (3.5) we deduce

$$\int_{A_k}\mu\,w+\frac14\int_{A_k}|Dw|^p+\frac14\int_{A_k}|Dw|^2(|Du_\varepsilon|^{p-2}+|Dv|^{p-2})\le c_\varepsilon\int_{A_k}(w^2+w^{p'})+|Dw|^pw\,.$$

Choosing k close to m_ε, since $w\le(m_\varepsilon-k)\to0$ and since $2\ge p'>1$, all the three terms in the right-hand side can be absorbed from the left. We deduce that $\frac12\int_{A_k}\mu\,w\le0$, that implies that $u_\varepsilon-v\le k<m_\varepsilon$, that yields to a contradiction. □

Let us stress some model case in which the above result can be applied. To fix the ideas, we consider the nonlinear elliptic equation

$$\lambda u-\Delta_p u+H(x,Du)=f(x)\qquad\text{in }\Omega\tag{3.6}$$

with $\lambda\ge0$, $H(x,\xi):\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ a Carathédory function such that $H(x,0)=0$, and $f\in L^1(\Omega)$.

In fact, we prove the comparison principle for general Hamiltonian terms $H(x,\xi)$ that behave as $\beta|\xi|^q$, with $q\in(p-1,p]$, up to a locally Lipschitz term that grows at most as $|\xi|^{\frac p2}$.

Corollary 3.3. *Let u, v be $W^{1,p}(\Omega)\cap L^\infty(\Omega)$ sub and supersolutions to (3.6) respectively. Assume that there exist $\beta > 0$ and $q \in (p-1, p]$, such that*

(3.7)
$$\Gamma(x,\xi) := H(x,\xi) - \beta|\xi|^q \quad \textit{satisfies}$$
$$|\Gamma(x,\xi) - \Gamma(x,\eta)| \leq \gamma \max\left\{1, |\xi|^{\frac{p-2}{2}} + |\eta|^{\frac{p-2}{2}}\right\}|\xi - \eta| \qquad \textit{for a.e. } x \in \Omega, \textit{ for any } \xi, \eta \in \mathbb{R}^N$$

for some $\gamma \geq 0$. Let one of the following assumptions be satisfied:

i) *$\lambda > 0$ and f is bounded below.*
ii) *$\lambda = 0$, f is bounded below and one between u or v is a strict subsolution or a strict supersolution, respectively;*
iii) *$\lambda = 0$ and $f \geq f_0 > 0$.*

If $u \leq v$ at $\partial\Omega$, then $u \leq v$ in Ω.

Proof. One can easily check that in the three cases (i)–(iii) the assumption (3.3) is satisfied, and in particular, when $\lambda = 0$, it holds with some $\alpha_\varepsilon > 0$ due to the strict sign of f (case (iii)) or to the strict character of the sub or super solution (case (ii)). □

Remark 3.4. Le us now summarize the picture of our results for the model case, namely for the equation
$$\lambda u - \Delta_p u + |Du|^q = f(x) \qquad \text{in } \Omega\,,$$
with $f \in L^1(\Omega)$, $p - 1 < q \leq p$ and $\lambda \geq 0$.

The results obtained in this Section allows us to deduce that the comparison principle between $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ sub/supersolutions holds true under the following sets of hypotheses:

- $\lambda = 0$ and $f(x) \geq 0$ (Corollary 3.1, applied with $g \equiv 1$);
- $\lambda > 0$, $f(x) \geq -c_0$ (Corollary 3.3 i), applied with $\Gamma \equiv 0$);
- $\lambda = 0$, $f(x) \geq -c_0$ and one between the sub and the super solution is strict (Corollary 3.3 ii), applied with $\Gamma \equiv 0$).

Moreover we stress that, if $f \in L^r(\Omega)$, $r > \frac{N}{p}$, then every nonnegative $u \in W_0^{1,p}(\Omega)$ which is a weak solution belongs to $L^\infty(\Omega)$.

4. Strong Comparison Principle

In this section we prove a strong comparison principle through the linearization of the p-Laplace operator. This approach was introduced by Damascelli and Sciunzi in [10, 11, 23, 24] in case of lower order terms depending on u. The main idea relies on the use of the weighted Sobolev space $W_\rho^{1,2}(\Omega)$, where $\rho = |Du|^{p-2}$; in this space it is possible to show (see [10]) weighted Sobolev and Poincaré inequalities provided the inverse of the weight satisfies a suitable summability property. Such a strategy has been extensively used to study qualitative properties of solutions, recent applications also include some examples of equations with first order terms, see e.g. [19]. It is then natural to see whether and how this strategy can be employed for the comparison principle with general first order terms. To fix the ideas, we consider the equation in the form

$$-\Delta_p u + H(x, Du) = f(x) \qquad \text{in } \Omega\,, \tag{4.1}$$

where $H \in C^1(\Omega \times \mathbb{R}^n)$ satisfies

$$H(x, 0) = 0\,, \tag{4.2}$$

which of course we may assume without loss of generality. The C^1 character of H could be relaxed into a local Lipschitz continuity (with respect to both variables), but this would require an additional approximation argument which we rather avoid here. It is more significant the following weighted Lipschitz condition that we require for H:

$$|H_\xi(x,\xi)| \le \gamma\,|\xi|^{\frac{p-2}{2}} \qquad \forall \xi\,:|\xi|\le 1 \qquad \text{for a.e. } x\in\Omega\,. \tag{4.3}$$

As far as the right-hand side is concerned, we will assume that

$$f\in W^{1,\infty}_{loc}(\Omega)\,,\qquad f(x)>0 \quad \text{in } \ x\in\Omega\,. \tag{4.4}$$

The strict sign on the right-hand side is a common assumption for the method of [10] to be applied; we refer to the above cited papers for a discussion of this issue, which is related to estimates of the degeneration set $\{x\in\Omega\ :\ Du(x)=0\}$. In the recent paper [24], it is actually shown that the strict sign condition could be relaxed by assuming that $f(x)$ might be zero at some point without exceeding some prescribed vanishing rate; the same kind of improvement would be possible here, nevertheless we confine ourselves to (4.4) for the sake of simplicity.

In the following, we use the local Lipschitz regularity of solutions of (4.1). We recall that, if $H(x,\xi)$ has natural growth, bounded solutions enjoy the $C^{1,\alpha}$ regularity in the interior of Ω (see [12, 25]) and up to the boundary (see [16]). Therefore, assuming the natural growth condition upon H would be enough for what follows.

We recall some basic definition and properties about the weighted Sobolev spaces (for more details about them see for instance [21, 26]). For $m\ge 1$ and $\rho\in L^1(\Omega)$ the weighted Sobolev space $W^{1,m}_\rho(\Omega)$ (with respect to the weight ρ) is defined as the set of functions $v\in L^m(\Omega)$ which are bounded with respect to the norm:

$$\|v\| = \left(\int_\Omega |v|^m\right)^{\frac1m} + \left(\int_\Omega |Dv|^m\rho\right)^{\frac1m}, \tag{4.5}$$

where Dv is the distributional derivative. According to [18], equivalently it is possible to define such a space as the completion of $C^\infty(\overline{\Omega})$ with respect to the norm defined in (4.5). As for the usual Sobolev spaces, the space $W^{1,m}_{0,\rho}$ is defined as the closure of $C^\infty_c(\Omega)$ in $W^{1,m}_\rho(\Omega)$. We set $H^1_\rho(\Omega)=W^{1,2}_\rho(\Omega)$ and $H^1_{0,\rho}(\Omega)=W^{1,2}_{0,\rho}(\Omega)$, which are the Hilbert spaces where the linearized operator associated to equation (4.1) is defined. Notice that, for $p>2$, the space $W^{1,p}(\Omega)$ is continuously embedded in $H^1_\rho(\Omega)$ if $\rho=|Du|^{p-2}$.

In the sequel we set $u_i=\frac{\partial u}{\partial x_i}$ and we denote by H_{x_i} the derivative of $H(x,\xi)$ with respect to the i-th component of the variable x and by H_ξ the gradient of $H(x,\xi)$ with respect to the second variable ξ. Moreover we set $Z=\{x\in\Omega : Du(x)=0\}$ and we denote by $B_r(x_0)$ the (open) ball of radius r and centered at x_0.

Let us now introduce the linearized equation associated to (4.1). Let $\varphi\in C^\infty_c(\Omega\setminus Z)$, and take φ_i, for any $i=1,...,n$, as test function in the weak formulation of (4.1). Integrating by parts, and since $u\in C^2(\Omega\setminus Z)$, we get that $u_i\in H^1_\rho(\Omega)$ is a solution to

$$L_u(u_i,\varphi)=0$$

if

$$\begin{aligned}&\int_\Omega |Du|^{p-2}\ Du_i\cdot D\varphi + (p-2)\int_\Omega |Du|^{p-4}(Du_i\cdot Du)(Du\cdot D\varphi)+\\ &+\int_\Omega H_{x_i}(x,Du)\varphi+\int_\Omega H_\xi(x,Du)\cdot Du_i\ \varphi-\int_\Omega f_i\varphi=0\,.\end{aligned} \tag{4.6}$$

By a density argument, (4.6) holds for any $\varphi\in H^1_\rho(\Omega)\cap L^\infty(\Omega)$ with compact support in $\Omega\setminus Z$.

4.1. **Weighted Sobolev and Poincaré inequalities.** This subsection is devoted to the proof of the following result.

Theorem 4.1. *Let $u \in W^{1,\infty}(\Omega)$ be a solution of (4.1), and assume (4.2)–(4.4) hold true. Then for every open set $\omega \subset\subset \Omega$, there exists a constant c_ω such that*

$$\|w\|_{L^2(\omega)} \le c_\omega \|Dw\|_{L^2_\rho(\omega)} \qquad \forall w \in H^1_{0,\rho}(\omega)$$

where $\rho = |Du|^{p-2}$. Moreover, we also have

$$\|w\|_{L^q(\omega)} \le c_\omega \|Dw\|_{L^2_\rho(\omega)} \qquad \forall w \in H^1_{0,\rho}(\omega)$$

for every $q < \frac{2N(p-1)}{N(p-1)-2}$.

In order to get these weighted Sobolev and Poincaré inequalities, it is required that the inverse of the weight enjoys some summability property. To get this we need an integral estimate on the Hessian of u, which is obtained using (4.6). We remark that the strict positivity of f is not required for the estimate of the Hessian, while we will use it to prove the summability of the inverse of the weight.

Observe that such result actually holds for any $p > 1$.

We start by proving the estimate on the Hessian of u.

Lemma 4.2. Let $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ be a weak solution of (4.1) and assume that $p \in (1, \infty)$. Let $x_0 \in \Omega$ and $r > 0$ be such that $B_{2r}(x_0) \subset \Omega$. For $\beta \in [0, 1)$ and $\gamma < N - 2$ ($\gamma = 0$ if $N = 2$), there holds:

$$\sup_{y\in\Omega} \int_{B_r(x_0)} \frac{|Du|^{p-2-\beta}|D^2u|^2}{|x-y|^\gamma} dx \le C\,, \tag{4.7}$$

where $C = C(x_0, r, \beta, \gamma, p, N, \|u\|_{W^{1,\infty}(\Omega)}, f, H)$ and $|D^2u|^2 = \displaystyle\sum_{i,j=1}^{n} |u_{ij}|^2$.

Proof. Let $G_\varepsilon : \mathbb{R} \to \mathbb{R}$ be defined as:

$$G_\alpha(s) = \begin{cases} s & \text{if } |s| \ge 2\alpha, \\ 2[s - \alpha \text{ sign}(s)] & \text{if } \alpha < |s| < 2\alpha, \\ 0 & \text{if } |s| \le \alpha, \end{cases}$$

and let ψ be a cut-off function such that

$$\psi \in C^\infty_c(B_{2r}(x_0)) \quad \psi \equiv 1 \text{ in } B_r(x_0) \qquad \text{and} \qquad |D\psi| \le \frac{2}{r}, \tag{4.8}$$

with $2r < \text{dist}(x_0, \partial\Omega)$. Fix $\beta \in [0, 1)$ and $\gamma < N - 2$ (or $\gamma = 0$ if $N = 2$) and set:

$$\varphi(x) = T_\varepsilon(u_i(x))K_\delta(|x-y|)\psi(x)^2 \quad \text{where} \quad T_\varepsilon(t) = \frac{G_\varepsilon(t)}{|t|^\beta} \quad \text{and} \quad K_\delta(t) = \frac{G_\delta(t)}{|t|^{\gamma+1}}\,. \tag{4.9}$$

Observe that φ is an admissible test function in (4.6), since it belongs to $H^1_\rho(\Omega)\cap L^\infty(\Omega)$, and we get:

$$
\begin{aligned}
&\int_\Omega |Du|^{p-4}\Big[|Du|^2|Du_i|^2+(p-2)(Du_i\cdot Du)^2\Big]T'_\varepsilon(u_i)K_\delta(|x-y|)\psi^2\\
&+\int_\Omega |Du|^{p-4}\Big[|Du|^2\ Du_i+(p-2)(Du_i\cdot Du)Du\Big]\cdot DK_\delta(|x-y|)\ T_\varepsilon(u_i)\psi^2\\
&+2\int_\Omega |Du|^{p-4}\Big[|Du|^2\ Du_i+(p-2)(Du_i\cdot Du)Du\Big]\cdot D\psi\ T_\varepsilon(u_i)K_\delta(|x-y|)\psi\\
&+\int_\Omega \big[H_{x_i}(x,Du)+(H_\xi(x,Du)\cdot Du_i)\big]\ T_\varepsilon(u_i)K_\delta(|x-y|)\psi^2\\
&=\int_\Omega f_iT_\varepsilon(u_i)K_\delta(|x-y|)\psi^2\,.
\end{aligned}
\tag{4.10}
$$

In the sequel, c, as well as C, will denote positive constants, possibly depending on $r, x_0, \|u\|_{W^{1,\infty}(\Omega)}$ but not on y, whose value can vary from line to line. First, observe that for any $p>1$ we have

$$|Du|^2|Du_i|^2+(p-2)(Du\cdot Du_i)^2\geq \min\{1,p-1\}|Du|^2|Du_i|^2$$

so that (4.10) implies

$$
\begin{aligned}
&\int_\Omega |Du|^{p-2}|Du_i|^2T'_\varepsilon(u_i)K_\delta(|x-y|)\psi^2\\
\leq\ & c\int_\Omega |Du|^{p-2}|Du_i||DK_\delta(|x-y|)||T_\varepsilon(u_i)|\psi^2\\
+\ & c\int_\Omega |Du|^{p-2}|Du_i||D\psi||T_\varepsilon(u_i)|K_\delta(|x-y|)\psi\\
+\ & c\int_\Omega \Big[|H_{x_i}(x,Du)|+|H_\xi(x,Du)||Du_i|\Big]|T_\varepsilon(u_i)|K_\delta(|x-y|)\psi^2\\
+\ & c\int_\Omega |f_i||T_\varepsilon(u_i)|K_\delta(|x-y|)\psi^2\,.
\end{aligned}
\tag{4.11}
$$

Recalling that $\gamma<N-2$ and that, for $s<N$, $\displaystyle\int_\Omega \frac{1}{|x-y|^s}dx$ is uniformly bounded (because Ω is bounded), for fixed $\varepsilon>0$ we can use dominate convergence to send δ to 0 and we get:

$$
\begin{aligned}
&\int_\Omega \frac{|Du|^{p-2}|Du_i|^2T'_\varepsilon(u_i)\psi^2}{|x-y|^\gamma}\\
\leq\ & c\int_\Omega \frac{|Du|^{p-2}|Du_i||T_\varepsilon(u_i)|\psi^2}{|x-y|^{\gamma+1}} && (I_1)\\
+\ & c\int_\Omega \frac{|Du|^{p-2}|Du_i||D\psi||T_\varepsilon(u_i)|\psi}{|x-y|^\gamma} && (I_2)\\
+\ & c\int_\Omega \frac{\Big[|H_{x_i}(x,Du)|+|f_i|\Big]|T_\varepsilon(u_i)|\psi^2}{|x-y|^\gamma} && (I_3)\\
+\ & c\int_\Omega \frac{|H_\xi(x,Du)||Du_i||T_\varepsilon(u_i)|\psi^2}{|x-y|^\gamma}\,. && (I_4)
\end{aligned}
\tag{4.12}
$$

Notice that the term (I_1) does not appear in the case $N=2$, since $DK_\delta\to 0$ if $\gamma=0$.

Using that $|T_\varepsilon(u_i)|\leq|u_i|^{1-\beta}$ (with $\beta\in[0,1)$) and since both f and H are locally Lipschitz continuous, by the regularity of u we immediately get that:

$$I_3\leq c\int_\Omega \frac{1}{|x-y|^\gamma}dx \tag{4.13}$$

and, since $\gamma < N-2$, it follows $I_3 \le c$.

Using Young inequality

$$\forall a, b \in \mathbb{R}, \forall \theta > 0 \qquad ab \le \theta a^2 + \frac{1}{4\theta} b^2 .$$

we have:

$$I_1 \le \vartheta \int_\Omega \frac{|Du|^{p-2}|Du_i|^2}{|x-y|^\gamma |u_i|^\beta} \frac{G_\varepsilon(u_i)}{u_i} \psi^2 + c, \tag{4.14}$$

where we recall that G_ε is an odd function and $\gamma < N-2$ (in the case $N=2$ and $\gamma = 0$ we had $I_1 = 0$). Recalling that $|D\psi| \le \frac{2}{r}$, we have:

$$I_2 \le \frac{\vartheta}{3} \int_\Omega \frac{|Du|^{p-2}|Du_i|^2}{|x-y|^\gamma |u_i|^\beta} \frac{G_\varepsilon(u_i)}{u_i} \psi^2 + c. \tag{4.15}$$

Using assumption (4.3), we also have:

$$I_4 \quad \le \quad c \int_\Omega \frac{|Du|^{\frac{p-2}{2}} |Du_i||T_\varepsilon(u_i)|\psi^2}{|x-y|^\gamma} \le \frac{\theta}{3} \int_\Omega \frac{|Du|^{p-2}|Du_i|^2}{|x-y|^\gamma |u_i|^\beta} \frac{G_\varepsilon(u_i)}{u_i} \psi^2 + c\,. \tag{4.16}$$

Recalling that for any $s > 0$, $T'_\varepsilon(s) = \frac{1}{|s|^\beta} \left[G'_\varepsilon(s) - \beta \frac{G_\varepsilon(s)}{s} \right]$, taking into account all the above estimates, we get:

$$\int_\Omega \frac{|Du|^{p-2}|Du_i|^2}{|u_i|^\beta |x-y|^\gamma} \left(G'_\varepsilon(u_i) - (\beta + \vartheta) \frac{G_\varepsilon(u_i)}{u_i} \right) \psi^2 \le c \tag{4.17}$$

We choose ϑ such that $\beta + \vartheta < 1$, so that $G'_\varepsilon(u_i) - (\beta+\vartheta)\frac{G_\varepsilon(u_i)}{u_i}$ is positive. By definition of G_ε, it follows that, $\forall s > 0$, $G'_\varepsilon(s) - (\beta+\vartheta)\frac{G_\varepsilon(s)}{s}$ tends to $1 - (\beta + \theta)$ as ε goes to 0 and hence by Fatou's Lemma we get:

$$\int_{\Omega \setminus \{u_i = 0\}} \frac{|Du|^{p-2}|Du_i|^2}{|u_i|^\beta |x-y|^\gamma} \psi^2 \le c\,. \tag{4.18}$$

Since $|u_i| \le |Du|$, it follows:

$$\int_{\Omega \setminus \{u_i = 0\}} \frac{|Du|^{p-2-\beta}|Du_i|^2}{|x-y|^\gamma} \psi^2 \le c\,,$$

and finally, since $u_i \in C^1(\Omega \setminus Z)$ and $Du_i = 0$ a.e. in $\{u_i = 0\} \cap (\Omega \setminus Z)$, we can conclude that

$$\int_{\Omega \setminus Z} \frac{|Du|^{p-2-\beta}|Du_i|^2}{|x-y|^\gamma} \psi^2 \le c\,, \tag{4.19}$$

where we remark that c depends on $x_0, r, N, p, \beta, \gamma, f, H, \|u\|_{W_0^{1,p}(\Omega)\infty}$ but it does not depend on y. Recalling the properties of ψ, we proved that

$$\sup_{y \in \Omega} \int_{B_r(x_0) \setminus Z} \frac{|Du|^{p-2-\beta}|D^2 u|^2}{|x-y|^\gamma} \le c\,.$$

Note however that, using the above computations with $\gamma = 0$, we can deduce from (4.17) that the function $|Du|^{\frac{p-\beta}{2}}$ belongs to $H^1_{loc}(\Omega)$. Since $p - \beta > 0$, this function vanishes in Z, so that, by standard properties of functions in Sobolev spaces, its gradient also vanishes a.e. in Z. This allows us to extend estimate (4.19) upon Z if we identify $|Du|^{p-2-\beta}|Du_i|^2$ as $|\frac{2}{p-\beta}\partial_i |Du|^{\frac{p-\beta}{2}}|^2$. This explains the statement (4.7) which we finally prove. □

Now, following [10], we prove an estimate on the inverse of the gradient of u.

Proposition 4.3. Let $u \in W^{1,p}_{\rm loc}(\Omega)$ be a solution of (4.1), $t \in [0, p-1)$, and $\gamma < N-2$ ($\gamma = 0$ if $N = 2$). Then, for any $\Omega' \subset\subset \Omega$ there exists c such that

$$\sup_{y\in\Omega} \int_{\Omega'} \frac{dx}{|Du|^t |x-y|^\gamma} \leq c \tag{4.20}$$

where $c = c(\Omega', t, \gamma, n, p, \|u\|_{W^{1,\infty}(\Omega)}, f, H)$.

Proof. We first prove inequality (4.20) on balls and then the thesis follows by a covering argument. For $x_0 \in \Omega$ we choose $r > 0$ such that $B_{2r}(x_0)$ is contained in Ω. We recall that the weak form of (4.1) is:

$$\int_\Omega |Du|^{p-2}\ Du \cdot D\varphi + \int_\Omega H(x, Du)\varphi = \int_\Omega f\varphi\,, \quad \forall \varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)\,. \tag{4.21}$$

Observe that, since $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$, actually one can take in (4.21) φ to be a function in $H^1(\Omega) \cap L^\infty(\Omega)$ with compact support.

Let ψ be the cut-off function defined above and consider

$$\varphi = \frac{1}{|Du|^t + \varepsilon} K_\delta(|x-y|)\psi^2\,, \qquad \text{where} \quad t = p-2+\beta < p-1$$

where K_δ has been defined in Lemma 4.2 and $\beta < 1$. We use φ in (4.21) and, recalling (4.4), we get:

$$\begin{aligned}
\alpha \int_{B_{2r}(x_0)} \frac{1}{|Du|^t+\varepsilon} K_\delta(|x-y|)\psi^2 &\leq \int_{B_{2r}(x_0)} f \frac{1}{|Du|^t+\varepsilon} K_\delta(|x-y|)\psi^2 \\
&= -\int_{B_{2r}(x_0)} \frac{t|Du|^{t-2}}{(|Du|^t+\varepsilon)^2} |Du|^{p-2} Du \cdot D^2u Du K_\delta(|x-y|)\psi^2 \\
+\int_{B_{2r}(x_0)} \frac{|Du|^{p-2} Du \cdot DK_\delta(|x-y|)}{|Du|^t+\varepsilon}\psi^2 &+ 2\int_{B_{2r}(x_0)} \frac{|Du|^{p-2} Du \cdot D\psi}{|Du|^t+\varepsilon} K_\delta(|x-y|)\psi \\
&+ \int_{B_{2r}(x_0)} H(x, Du) \frac{1}{|Du|^t+\varepsilon} K_\delta(|x-y|)\psi^2
\end{aligned}$$

where α is a positive lower bound of f in $\Omega' + 2r$, $2r < \mathrm{dist}(\Omega', \partial\Omega)$. We infer:

(4.22)

$$\begin{aligned}
\int_{B_{2r}(x_0)} \frac{1}{|Du|^t+\varepsilon} K_\delta(|x-y|)\psi^2 &\leq \frac{t}{\alpha}\int_{B_{2r}(x_0)} \frac{|Du|^t}{(|Du|^t+\varepsilon)^2} |Du|^{p-2} |D^2u| K_\delta(|x-y|)\psi^2 \\
+\frac{1}{\alpha}\int_{B_{2r}(x_0)} \frac{|Du|^{p-1}|DK_\delta(|x-y|)|}{|Du|^t+\varepsilon}\psi^2 &+ \frac{2}{\alpha}\int_{B_{2r}(x_0)} \frac{|Du|^{p-1}|D\psi|}{|Du|^t+\varepsilon} K_\delta(|x-y|)\psi \\
&+\frac{1}{\alpha}\int_{B_{2r}(x_0)} \frac{|H(x, Du)|}{|Du|^t+\varepsilon} K_\delta(|x-y|)\psi^2\,.
\end{aligned}$$

Arguing as above, we use dominate convergence theorem sending δ to 0 and we get:

$$
\begin{aligned}
&\int_{B_{2r}(x_0)} \frac{1}{|Du|^t+\varepsilon}\frac{1}{|x-y|^\gamma}\psi^2 \\
\leq\ & \frac{t}{\alpha}\int_{B_{2r}(x_0)} \frac{|Du|^t}{(|Du|^t+\varepsilon)^2}|Du|^{p-2}|D^2u|\frac{1}{|x-y|^\gamma}\psi^2 && (J_1)\\
+\ & \frac{1}{\alpha}\int_{B_{2r}(x_0)} \frac{|Du|^{p-1}}{|Du|^t+\varepsilon}\frac{1}{|x-y|^{\gamma+1}}\psi^2 && (J_2)\\
+\ & \frac{2}{\alpha}\int_{B_{2r}(x_0)} \frac{|Du|^{p-1}|D\psi|}{|Du|^t+\varepsilon}\frac{1}{|x-y|^\gamma}\psi && (J_3)\\
+\ & \frac{1}{\alpha}\int_{B_{2r}(x_0)} \frac{|H(x,Du)|}{|Du|^t+\varepsilon}\frac{1}{|x-y|^\gamma}\psi^2\,. && (J_4)
\end{aligned}
$$

Recalling the definition of t, we have that

$$
\begin{aligned}
J_1 &\leq \theta\int_{B_{2r}(x_0)} \frac{1}{(|Du|^t+\varepsilon)|x-y|^\gamma}\psi^2 + \frac{1}{4\theta}\int_{B_{2r}(x_0)} \frac{|Du|^{p-2-\beta}|D^2u|^2}{|x-y|^\gamma}\psi^2\\
&\leq \theta\int_{B_{2r}(x_0)} \frac{1}{(|Du|^t+\varepsilon)|x-y|^\gamma}\psi^2 + \frac{c}{\theta}\,,
\end{aligned}
\tag{4.23}
$$

for $\theta > 0$, where we used Lemma 4.2 to estimate the second term in the right hand side. Since $t < p-1$ we deduce that $\frac{|Du|^{p-1}}{|Du|^t+\varepsilon}$ is bounded and consequently

$$
J_2 \leq c\int_{B_{2r}(x_0)} \frac{1}{|x-y|^{\gamma+1}}\psi^2 \leq c\,. \tag{4.24}
$$

By assumption on ψ we get,

$$
J_3 \leq \frac{c}{r}\int_{B_{2r}(x_0)} \frac{1}{|x-y|^\gamma}\psi^2 \leq c\,. \tag{4.25}
$$

while by (4.3), thanks to Young inequality

$$
\begin{aligned}
J_4 &= \int_{B_{2r}(x_0)} \frac{|H(x,Du)|}{|Du|^t+\varepsilon}\frac{1}{|x-y|^\gamma}\psi^2\\
&\leq \theta\int_{B_{2r}(x_0)} \frac{1}{|Du|^t+\varepsilon}\frac{1}{|x-y|^\gamma}\psi^2 + \frac{c}{\theta}\int_{B_{2r}(x_0)} \frac{1}{|x-y|^\gamma}\psi^2
\end{aligned}
\tag{4.26}
$$

We choose θ small enough and by (4.23)–(4.26) we get:

$$
\int_{B_{2r}(x_0)} \frac{1}{|Du|^t+\varepsilon}\frac{1}{|x-y|^\gamma}\psi^2 \leq c \tag{4.27}
$$

and the thesis follows using the Fatou's Lemma as ε tends to 0. □

The estimates proved in Lemma 4.2 and Proposition 4.3 are crucial to prove the weighted Sobolev and Poincaré inequalities. More precisely, we deduce from [10] the following consequence.

Proof. [of Theorem 4.1] See Theorem 3.1 in [10]. In such a paper it is proved that for a weight ρ satisfying (4.20) a Sobolev inequality holds with q given by $\frac{1}{q} = \frac{1}{2} - \frac{1}{N} + \frac{1}{2t}(1-\frac{\gamma}{N})$. Since, by Proposition 4.3, the weight $\rho = |Du|^{\frac{p-1}{p-2}}$ satisfies this estimate for every $\gamma < N-2$ and $t > \frac{p-1}{p-2}$, we deduce the bound for q given in the statement. □

4.2. **Weak Harnack Inequality and Strong Comparison Principle.** In order to deal with the *Strong Comparison Princple* it is enough to apply the weighted estimates proved above and to follow the classical Moser iteration method ([20], [13], [26]); this was another remarkable consequence of the approach suggested by L.Damascelli and D.Sciunzi (see [11], Appendix A).

The result obtained is the following.

Theorem 4.4 (Weak Harnack Inequality). *Let $u, v \in W^{1,\infty}_{loc}(\Omega)$. Assume that either u or v is a weak solution to (4.1) and $H(x,\xi)$, f satisfying (4.3)–(4.4). Assume that*

$$-\Delta_p u + H(x, Du) \le -\Delta_p v + H(x, Dv)\,, \qquad u \le v \quad in \;\; B_{5\delta}(x_0) \tag{4.28}$$

for some $x_0 \in \Omega$, $\delta > 0$ such that $\overline{B_{5\delta}(x_0)} \subset \Omega$. Then

$$\|v - u\|_{L^q(B_{2\delta}(x_0))} \le c \inf_{B_\delta(x_0)} (v - u) \tag{4.29}$$

where c does not depend on δ and

$$\frac{1}{q} > \frac{1}{2} - \frac{1}{N} + \frac{1}{N}\Big(\frac{p-2}{p-1}\Big)\,.$$

Proof. The proof of such a result follows the same outline of Theorem 3.3 in [11]. Two preliminary estimates for the difference $u - v$ are needed in order to apply the Moser's iterative method. We avoid to repeat this latter step, that is developed in all its details in [11], Appendix A. Thus we only concentrate on the two basic integral estimates which allows one to start the iterations.

Let us denote by z be the difference between v and u: observe that $z = v - u \ge 0$ and we can make such an inequality strict by adding a positive constant to v.

Estimate 1. Let us multiply (4.28) by $\eta^2(v-u)^\beta$, for $\beta < 0$ and $\eta \in C^1_c(\Omega)$ being a cut-off function. Integrating by parts, and denoting $\rho = |Dv|^{p-2} + |Du|^{p-2}$, we have

$$\begin{aligned}|\beta| \int_\Omega \Big(|Dv|^{p-2}Dv - |Du|^{p-2}Du\Big) Dz\ \eta^2 z^{\beta-1} \\ \le 2\int_\Omega \rho\ |Dz||D\eta|\ \eta z^\beta + \int_\Omega \Big|H(x,Du) - H(x,Dv)\Big|\eta^2 z^\beta\,.\end{aligned}$$

Now, by (4.3) and using Young inequality we have

$$\begin{aligned}\int_\Omega \Big|H(x,Du) - H(x,Dv)\Big|\eta^2 z^\beta &\le c\int_\Omega (|Dv|^{\frac{p-2}{2}} + |Du|^{\frac{p-2}{2}})|Dz|\eta^2 z^\beta \\ &\le \frac{|\beta|}{2}\int_\Omega \rho\,|Dz|^2\eta^2 z^{\beta-1} + \frac{c}{|\beta|}\int_\Omega \eta^2 z^{\beta+1}\,.\end{aligned}$$

Hence, using the monotonicity of p-Laplacian, we get

$$\frac{|\beta|}{2}\int_\Omega \rho\,|Dz|^2\eta^2 z^{\beta-1} \le 2\int_\Omega \rho\ |Dz||D\eta|\ \eta z^\beta + \frac{c}{|\beta|}\int_\Omega \eta^2 z^{\beta+1}$$

which yields

$$\int_\Omega \rho\,|Dz|^2\ \eta^2 z^{\beta-1} \le \frac{c}{\beta^2}\int_\Omega \Big[\rho|D\eta|^2 + \eta^2\Big] z^{\beta+1}\,. \tag{4.30}$$

Estimate 2. Let us take now $\gamma > 1$ and, as before, we denote by $z = v - u > 0$ (where if it is only nonnegative, we add a positive constant). Then we choose $\eta^2 \frac{1}{z}\big(\big|\log z\big|^\gamma + 2(2\gamma)^\gamma\big)$ as test function, obtaining

$$\begin{aligned}&\int_\Omega \Big(|Dv|^{p-2}Dv-|Du|^{p-2}Du\Big)\frac{Dz}{z^2}\ \eta^2\big(\big|\log z\big|^\gamma+2(2\gamma)^\gamma-\gamma\big|\log z\big|^{\gamma-2}\log z\big)\\ &\quad\le \int_\Omega 2\eta\Big||Dv|^{p-2}Dv-|Du|^{p-2}Du\Big|\frac{|D\eta|}{z}\ \big(\big|\log z\big|^\gamma+2(2\gamma)^\gamma\big)\\ &\qquad+\int_\Omega\Big|H(x,Du)-H(x,Dv)\Big|\frac{1}{z}\ \eta^2\big(\big|\log z\big|^\gamma+2(2\gamma)^\gamma\big)\,.\end{aligned}$$

Notice that, by Young's inequality, we have $2\gamma|\log z|^{\gamma-1}\le(1-\frac{1}{\gamma})|\log z|^\gamma+\frac{1}{\gamma}(2\gamma)^\gamma$, so

$$\big|\log z\big|^\gamma+2(2\gamma)^\gamma-\gamma\big|\log z\big|^{\gamma-2}\log z\ge\frac{1}{\gamma}\big|\log z\big|^\gamma+(2\gamma)^\gamma+\gamma|\log z|^{\gamma-1}\,. \tag{4.31}$$

Therefore, using the monotonicity of p-Laplacian in the left-hand side we get

$$\begin{aligned}&\int_\Omega\rho\,\frac{|Dz|^2}{z^2}\ \eta^2\big(\frac{1}{\gamma}\big|\log z\big|^\gamma+(2\gamma)^\gamma+\gamma|\log z|^{\gamma-1}\big)\\ &\quad\le \int_\Omega 2\eta\Big||Dv|^{p-2}Dv-|Du|^{p-2}Du\Big|\frac{|D\eta|}{z}\ \big(\big|\log z\big|^\gamma+2(2\gamma)^\gamma\big)\\ &\qquad+\int_\Omega\Big|H(x,Du)-H(x,Dv)\Big|\frac{1}{z}\ \eta^2\big(\big|\log z\big|^\gamma+2(2\gamma)^\gamma\big)\,.\end{aligned}$$

As before, we estimate

$$\begin{aligned}&\int_\Omega\Big|H(x,Du)-H(x,Dv)\Big|\frac{1}{z}\ \eta^2\big(\big|\log z\big|^\gamma+2(2\gamma)^\gamma\big)\\ &\quad\le\frac12\int_\Omega\rho\,\frac{|Dz|^2}{z^2}\eta^2\big(\frac{1}{\gamma}\big|\log z\big|^\gamma+(2\gamma)^\gamma\big)+c\,\gamma\int_\Omega\eta^2\big(\big|\log z\big|^\gamma+(2\gamma)^\gamma\big)\end{aligned}$$

and similarly

$$\begin{aligned}&\int_\Omega 2\eta\Big||Dv|^{p-2}Dv-|Du|^{p-2}Du\Big|\frac{|D\eta|}{z}\ \big(\big|\log z\big|^\gamma+2(2\gamma)^\gamma\big)\\ &\quad\le\frac12\int_\Omega\rho\,\frac{|Dz|^2}{z^2}\eta^2\big(\frac{1}{\gamma}\big|\log z\big|^\gamma+(2\gamma)^\gamma\big)+c\gamma\int_\Omega\rho|D\eta|^2\,\big(\big|\log z\big|^\gamma+(2\gamma)^\gamma\big)\end{aligned}$$

Therefore, overall we deduce, denoting $w=\log z$ and noticing that $\frac{|Dz|^2}{z^2}=|Dw|^2$,

$$\int_\Omega\rho\ \eta^2|Dw|^2w^{\gamma-1}\le c\int_\Omega[\rho|D\eta|^2+\eta^2](w^{\gamma+1}+(2\gamma)^\gamma)\,. \tag{4.32}$$

Once that (4.30) and (4.32) hold, we follow [11] (Appendix A) in order to apply the iteration method. Hence we can construct a family of cut-off functions η supported in $B_{5\delta}(x_0)$ such that $\eta\equiv1$ in $B_\delta(x_0)$ that finally let us prove that (4.29) holds true. □

Finally, the result of Theorem 1.4 is a straightforward corollary of the previous weak Harnack inequality.

Appendix A. A direct proof of the comparison principle in the Lipschitz framework

In this appendix we deal with the comparison principle for solutions of the equation

$$\lambda z-\Delta_p z+H(x,Dz)=0\qquad\text{in }\Omega \tag{A.1}$$

where either the solution or the Hamiltonian term are Lipschitz.

We summarize our result in the following Theorem.

Theorem A.1. *Let u and v be a subsolution and a supersolution, respectively, to the equation (A.1) with $\lambda > 0$. Assume that one of the two following conditions holds true*

(i) *$u, v \in W^{1,p}(\Omega)$ and $\xi \mapsto H(x, \xi)$ is Lipschitz continuous, i.e.*

$$\exists c > 0 : \quad |H(x,\xi) - H(x,\eta)| \le c|\xi - \eta| \qquad \text{for a.e. } x \in \Omega\,,\ \forall \xi, \eta \in \mathbb{R}^N\,; \tag{A.2}$$

(ii) *$u, v \in W^{1,\infty}(\Omega)$ and $\xi \mapsto H(x,\xi)$ is locally Lipschitz continuous.*

If $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.

Remark A.2. One may think that the comparison principle for Lipschitz solutions could as well be derived by general results of viscosity solutions theory; on the other hand, we are not assuming here any continuity assumption of H with respect to x, so (A.1) might not be formulated in viscosity sense.

The proof of the above result is based on the following easy, but useful, principle that gives the comparison through a suitable direct choice of test functions in the weak formulation.

Proposition A.3. *Let z_1 and z_2 be $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ functions satisfying*

$$\lambda(z_1 - z_2) - \Delta_p z_1 + \Delta_p z_2 \le \beta|D(z_1 - z_2)| + \gamma|D(z_1 - z_2)|^q \qquad \text{in } \Omega\,, \tag{A.3}$$

for some $\lambda > 0, \beta, \gamma \ge 0$ and $1 < q \le p$. If $z_1 \le z_2$ on $\partial\Omega$, then $z_1 \le z_2$ in Ω.

Moreover, if $\gamma = 0$ the conclusion holds assuming only $z_1, z_2 \in W^{1,p}(\Omega)$.

Before proving such a result, let us introduce, for any $\delta > 0$, the following function

$$\begin{cases} \psi_\delta(s) = e^{-\frac{1}{\delta}\frac{1}{p-2}\frac{1}{s^{p-2}}} & \text{if } s > 0 \\ \psi_\delta(0) = 0\,, \end{cases} \tag{A.4}$$

which is the unique solution to the Cauchy problem

$$\begin{cases} \delta s^{p-1}\psi_\delta'(s) = \psi_\delta(s) & s \in (0,+\infty) \\ \psi_\delta(0) = 0\,. \end{cases} \tag{A.5}$$

Notice that ψ is a $C^1[0,+\infty)$, positive, bounded and increasing function.

Proof. According to the hypotheses, the function $z = z_1 - z_2$ belongs to $W^{1,p}(\Omega) \cap L^\infty(\Omega)$; let us choose $\psi_\delta(z^+)$ as test function in the weak formulation of (A.3), with a suitable choice of δ to be fixed later (depending on p, q, γ, β and $\|z\|_{L^\infty(\Omega)}$). Notice that this is admissible since $\psi_\delta \in C^1[0,+\infty)$ and $z^+ = 0$ at $\partial\Omega$. Hence we deduce that

$$\int_\Omega \lambda z \psi_\delta(z^+) + \int_\Omega \psi_\delta'(z^+)|Dz|^p \le \int_\Omega \beta\psi_\delta(z^+)|Dz| + \int_\Omega \gamma\psi_\delta(z^+)|Dz|^q\,,$$

and by (A.5) we have:

$$\lambda \int_\Omega z\psi_\delta(z^+) + \frac{1}{\delta}\int_\Omega \psi_\delta(z^+)\frac{|Dz|^p}{z^{p-1}} \le \beta\int_\Omega \psi_\delta(z^+)|Dz| + \gamma\int_\Omega \psi_\delta(z^+)|Dz|^q\,. \tag{A.6}$$

Now we apply Young inequality on the right hand side in the following way:

$$\beta|Dz| = \beta\Big(\frac{4}{\lambda p'}\Big)^{\frac{1}{p'}}\frac{|Dz|}{z^{\frac{p-1}{p}}}\Big(\frac{\lambda p'}{4}\Big)^{\frac{1}{p'}} z^{\frac{p-1}{p}} \le \frac{1}{p}\beta^p\Big(\frac{4}{\lambda p'}\Big)^{p-1}\frac{|Dz|^p}{z^{p-1}} + \frac{\lambda}{4}z\,,$$

and

$$\gamma|Dz|^q = \gamma\frac{\|z\|^{q-1}_{L^\infty(\Omega)}}{\left(\frac{\lambda}{4}\frac{p}{p-q}\right)^{\frac{p-q}{p}}}\frac{|Dz|^q}{z^{\frac{q(p-1)}{p}}}\,\frac{\left(\frac{\lambda}{4}\frac{p}{p-q}\right)^{\frac{p-q}{p}}}{\|z\|^{q-1}_{L^\infty(\Omega)}}z^{\frac{q(p-1)}{p}} \le \frac{q}{p}\gamma^{\frac{p}{q}}\frac{\|z\|^{\frac{p}{q'}}_{L^\infty(\Omega)}}{\left(\frac{\lambda}{4}\frac{p}{p-q}\right)^{\frac{p-q}{q}}}\frac{|Dz|^p}{z^{p-1}} + \frac{\lambda}{4}z$$

(here we deal with the case $1 < q < p$, the case $q = p$ is direct) so that

$$\frac{\lambda}{2}\int_\Omega z\psi_\delta(z^+) + \frac{1}{\delta}\int_\Omega \frac{|Dz|^p}{z^{p-1}}\psi_\delta(z^+) \le \left[\frac{\beta^p}{p}\big(\frac{4}{\lambda p'}\big)^{p-1} + \frac{q}{p}\gamma^{\frac{p}{q}}\frac{\|z\|_{L^\infty(\Omega)}^{\frac{p}{q'}}}{\big(\frac{\lambda}{4}\frac{p}{p-q}\big)^{\frac{p-q}{q}}}\right]\int_\Omega \frac{|Dz|^p}{z^{p-1}}\psi_\delta(z^+)\,.$$

Choosing $\frac{1}{\delta}$ large enough, we deduce that $\displaystyle\int_\Omega z\psi_\delta(z^+) \le 0$ and consequently $z^+ \equiv 0$.

Notice that δ depends on $\|z\|_{L^\infty(\Omega)}$ only if $\gamma > 0$, otherwise there is no need to assume that the two functions are bounded. □

Remark A.4. By choosing $z_2 \equiv 0$, the above result shows that, if $\lambda > 0$, $\beta, \gamma \ge 0$ and $1 < q \le p$, any $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ subsolution of

$$\lambda z - \Delta_p z = \beta|Dz| + \gamma|Dz|^q \qquad \text{in } \Omega\,,$$

is non positive.

Remark that this conclusion is false, in general, if z is unbounded and $\gamma > 0$. On the other hand, as we mentioned in the statement, if $\gamma = 0$ the conclusion is true for merely $W^{1,p}$ subsolutions.

Proof. [of Theorem A.1]. If $w = u - v$, we have

$$\lambda(u - v) - \Delta_p u + \Delta_p v \le |H(x, Du) - H(x, Dv)|\,.$$

Using (A.2) (or its local version if solutions are Lipschitz), both in case (i) and (ii) the conclusion follows applying Proposition A.3 with $\gamma = 0$. □

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