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A finite difference scheme for the fractional Laplacian on non-uniform grids

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Abstract In this study, we analyze the convergence of the finite difference method on non-uniform grids and provide examples to demonstrate its effectiveness in approximating fractional differential equations involving the fractional Laplacian. y utilizing non-uniform grids, it becomes possible to achieve higher accuracy and improved resolution in specific regions of interest. Overall, our findings indicate that finite difference approximation on non-uniform grids can serve as a dependable and efficient tool for approximating fractional Laplacians across a diverse array of applications.

Keywords Fractional differential equations · Caputo fractional derivative · Fractional Laplacian · Finite difference method · meshless method

1 Introduction

In the last two decades, fractional diffusion operators have gained significant attention as replacements for the standard Laplace operator and other types of elliptic operators with variable coefficients. These new operators allow for the inclusion of long-range interactions in the theory, which are common in many applications. Unlike the standard operators, which operate through pointwise differentiation, fractional diffusion operators work through global integration using a singular kernel, indicating the nonlocal nature of the process. It is commonly observed that processes involving long-range interactions, such as those described in this paper, occur frequently in nature, as demonstrated in the biological observations. These types of processes also appear in various fields such as finance, fluid mechanics, solid state physics, and polymer chemistry, see [2] and [6] and the references therein.
The fractional Laplacian operators are often used to mathematically model

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anomalous diffusion, which refers to processes that involve interactions over longer distances than those of random walks or short-distance interactions found in Brownian motion. These operators can generate stable Lévy processes that involve jumps and long-distance interactions, and are useful for explaining anomalous diffusion in various fields such as continuum mechanics, phase transitions, population dynamics, and finance [26] and [27]. In this short communication we are concerned with the numerical treatment of the following fractional differential equation

$$\begin{cases} \epsilon \dfrac{\partial y}{\partial t} = (-\Delta)^{\frac{\gamma}{2}} y + \Psi(y) + c(y,t,x), & x \in [a,b], \quad t \in (0,\infty), \\ y(t,a) = y_a, \quad y(t,b) = y_b, & t \in (0,\infty), \\ y(0,x) = y^0(x), & x \in [a,b], \end{cases} \tag{1}$$

where ϵ, y_a and y_b are real numbers, c and y^0 are known functions and Ψ is

$$\Psi(\cdot) := d_1(\cdot,t,x)\frac{\partial}{\partial x}(\cdot) + d_2(\cdot,t,x)\frac{\partial^2}{\partial x^2}(\cdot) \tag{2}$$

The main idea of the paper is to obtain a discretization of the fractional Laplacian operator via its pointwise representation in terms of the Riesz derivative. More precisely, it is stated that ([14],[28])

$$-(-\Delta)^{\frac{\gamma}{2}} y(x) = {}_{RZ}\, D_x^{\gamma} y(x), \tag{3}$$

for $\gamma \in (0,1) \cup (1,2)$ and a suitably smooth function $y(x)$ define on $(a,b) \subset \mathbb{R}$.

Definition 1 The γ-th order Riesz derivative of is given by

$${}_{RZ}D_x^{\gamma} y(x) = -\frac{1}{2\cos(\frac{\gamma\pi}{2})}\left({}_aD_x^{\gamma} y(x) + {}_x D_b^{\gamma} y(x)\right), \tag{4}$$

Here, ${}_aD_x^{\gamma}y(x)$ and ${}_xD_b^{\gamma}y(x)$ denote the left and right Riemann-Liouville derivatives of y, respectively.

Definition 2 Let $n-1 < \gamma \leq n$ and $n \in \mathbb{N}$. The left and right Riemann-Liouville derivative are defined as

$$\begin{aligned} {}_aD_x^{\gamma} y &= \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dx^n}\int_a^x (x-s)^{n-\gamma-1} y ds, \\ {}_xD_b^{\gamma} y &= \frac{(-1)^n}{\Gamma(n-\gamma)}\frac{d^n}{dx^n}\int_a^x (s-x)^{n-\gamma-1} y ds, \end{aligned} \tag{5}$$

respectively.

We aim to obtain a discretization of (3), so we need to express the Riesz fractional derivative in terms of the left and right Caputo fractional derivatives. This way we can use the explicit formulae given in [25]. First, we recall the definitions of the left and right Caputo fractional derivatives:

Definition 3 Let $n-1<\gamma\leq n$ and $n\in\mathbb{N}$. The left and right Caputo fractional derivative are defined as

$$
\begin{aligned}
{}_{a}^{C}D_{x}^{\gamma}y &= \frac{1}{\Gamma(n-\gamma)}\int_{a}^{x}\frac{y^{(n)}}{(x-s)^{\gamma+1-n}}ds,\\
{}_{x}^{C}D_{b}^{\gamma}y &= \frac{(-1)^{n}}{\Gamma(n-\gamma)}\int_{x}^{b}\frac{y^{(n)}}{(s-x)^{\gamma+1-n}}ds,
\end{aligned}
\tag{6}
$$

respectively.

The following relations are important for our purpose [3]:

$$
\begin{aligned}
{}_{a}^{C}D_{x}^{\gamma}f(x) &= {}_{a}D_{x}^{\gamma}f(x)-\sum_{k=0}^{n-1}\frac{f^{k}(a)(x-a)^{k-\gamma}}{\Gamma(k-\gamma+1)},\\
{}_{x}^{C}D_{b}^{\gamma}f(x) &= {}_{x}D_{b}^{\gamma}f(x)-\sum_{k=0}^{n-1}\frac{f^{k}(b)(b-x)^{k-\gamma}}{\Gamma(k-\gamma+1)}.
\end{aligned}
\tag{7}
$$

Hence, by combining the previous expressions, one can use the following representation of the fractional Laplacian for $\gamma\in(0,1)\cup(1,2)$

$$
\begin{aligned}
(-\Delta)^{\frac{\gamma}{2}}y(x) =& \frac{1}{2\cos(\frac{\gamma\pi}{2})}\Bigg({}_{a}^{C}D_{x}^{\gamma}y(x)+{}_{x}^{C}D_{b}^{\gamma}y(x)\\
&+\sum_{k=0}^{n-1}\frac{y^{k}(a)(x-a)^{k-\gamma}+y^{k}(b)(b-x)^{k-\gamma}}{\Gamma(k-\gamma+1)}\Bigg),
\end{aligned}
\tag{8}
$$

and obtain the equivalent problem to (1):

$$
\begin{cases}
\epsilon\dfrac{\partial y}{\partial t}=\dfrac{1}{2\cos(\frac{\gamma\pi}{2})}({}_{a}^{C}D_{x}^{\gamma}y+{}_{x}^{C}D_{b}^{\gamma}y)+\Psi(y)+q(y,t,x), & x\in[a,b],t\in(0,\infty),\\
y(t,a)=y_{a},\quad y(t,b)=y_{b}, & t\in(0,\infty),\\
y(0,x)=y^{0}(x), & x\in[a,b],
\end{cases}
\tag{9}
$$

with

$$
q(y,t,x)=c(y,t,x)+\frac{1}{2\cos(\frac{\gamma\pi}{2})}\sum_{k=0}^{n-1}\frac{y^{k}(a)(x-a)^{k-\gamma}+y^{k}(b)(b-x)^{k-\gamma}}{\Gamma(k-\gamma+1)}.
$$

We propound a finite difference scheme following the procedure described in [25], based on the existence of the Taylor series for the Caputo derivative and moving least squares. We shall use the so-called sequential Caputo derivative given by

$$
{}_{a}^{C}D_{x}^{m\gamma}=\underbrace{{}_{a}^{C}D_{x}^{\gamma}\,{}_{a}^{C}D_{x}^{\gamma}\cdots{}_{a}^{C}D_{x}^{\gamma}}_{m\text{ times}}.
$$

Under this notation, the fractional Taylor series of a function $y(x)$ as in [4], [20] and [24]:

$$\begin{aligned} y(x) &= \sum_{r=0}^{m} {}_{a}^{C}D_{x}^{r\gamma} y(a) \frac{(x-a)^{r\gamma}}{\Gamma(r\gamma+1)} + R_m(x,a), \\ y(x) &= \sum_{r=0}^{m} {}_{x}^{C}D_{b}^{r\gamma} y(b) \frac{(b-x)^{r\gamma}}{\Gamma(r\gamma+1)} + R_m(x,b). \end{aligned} \tag{10}$$

As a consecuenque of its importance, there are various approaches to the numerical solution of fractional Laplacian problems, including finite difference, finite element, and spectral methods. These methods involve approximating the fractional Laplacian using a finite set of discrete points and then solving the resulting system of equations using numerical techniques. In [7], the authors analyze a nonlocal diffusion operator that has the fractional Laplacian and fractional differential operators as special cases, using a nonlocal vector calculus to define a weak formulation of the nonlocal problem with a continuous Galerkin finite element discretization. Many numerical methods relating the fractional Laplacian and fractional derivatives have been summarized in [8]. [9] presents a class of weighted finite difference methods (WFDMs) for solving a class of initial-boundary value problems of space fractional partial differential equations with variable coefficients, and examines their stability and convergence properties.

In [13], a finite difference method is introduced for solving a parabolic equation of the Caputo type with a fractional Laplacian, which is approximated using the Caffarelli-Silvestre extension. Similarly, in [22], the author propounded a finite difference scheme for the porous medium equation with the same extension. A combined finite difference with numerical quadrature method to obtain a discrete convolution operator with positive weights is employed to discretize the fractional Laplacian, based on the singular integral representationm in[14]. In [19], the authors use the first degree tensor product finite element. Finally, several methods such as the $L1/L2$-approximation, the standard/shifted Grünwald method, the matrix transform method (MTM) and the method of lines are applied in [28] for solving space Riesz derivatives. The previous list is not an exhaustive one, but just a good sample of the great interest that has been aroused in the numerical solution of differential equations involving anomalous diffusion terms, such as the fractional Laplacian.

The procedure described in this communication follows the idea of the Generalized Finite Difference Method (GFDM), which has attracted attention since the difference between two points is not necessarily fixed, but is instead defined by a weight function. The method is based on the technique of moving least squares, through which, by means of the optimization of the weighted Taylor polynomial (based on the distance between nodes) allows finding a discretization of the derivatives at a point based on the value of the nearby nodes, even if they are not evenly spaced. The main novelty of this work is to obtain a numerical approximation of a representation of the fractional Laplacian using the Moving Least Squares technique and the generalized Taylor polynomial for fractional derivatives. This enables, on one hand, the derivation of simple and useful expressions for the numerical schemes, and on the other hand, the application of these schemes to irregular domains with non-equidistant nodes. The generalized finite differences method was created in the 1970s with the work of Jensen [15], Perrone and Kao [21], although it was already suggested by Collatz [5] and Forsythe and Wasow [10] ten years earlier, and a large number of authors have contributed to its progress and improvement, as can be seen in [1]. The GFD formulae is particularly useful when the function being approximated has a singularity or some other features that makes it difficult to approximate using traditional finite differences. This method have been recently applied for solving equations involving fractional Laplacian in [11], where the Caffarelli-Silvestre extension and the GFDM are used.

This note is organized as follows: in the next section we find the explicit finite difference formulae for the first, second and fractional-order derivatives. In Section 3, convergence and stability results are obtained. Section 4 is devoted to showing the applicability and accuracy of the method with examples of evolution ($\epsilon \neq 0$) and stationary ($\epsilon = 0$) problems. Finally, we extract some conclusions.

2 Numerical approach

The approximation of the spatial integer-order derivatives is given by the GFD formulae, and its derivation can be found in [25]. Let $Z \subset [a,b]$ be a finite set of points and a subset of Z with p points

$$\{x_i : i = 1, ..., p, 1 \leq i \leq p\} \subset Z$$

with center at x_0. The way nodes are selected can vary (see [23] for more details).

Consider the fractional truncated Taylor expansion of y evaluated in x_i and centered at x_0 (as in [24] and [25])

$$y(x_i) = y(x_0) + {}_a^C D_x^{\gamma} y(x_0) \frac{\Lambda_1(x_i)}{\Gamma(\gamma+1)} + {}_a^C D_x^{2\gamma} y(x_0) \frac{\Lambda_2(x_i)}{\Gamma(2\gamma+1)} + R_2(x, x_0, a), \tag{11}$$

where

$$\Lambda_1(x_i) = x_i^{\gamma} - x_0^{\gamma}, \quad \Lambda_2(x_i) = x_i^{2\gamma} - x_0^{2\gamma} - \frac{\Gamma(2\gamma+1)}{\Gamma^2(\gamma+1)} x_0^{\gamma} \Lambda_1(x_i).$$

Let Y_i be the approximation of $y(x_i)$ and $\Lambda_j^i = \Lambda_j(x_i)$, together with

$$\mathbf{\Pi_Y} := ({}_a^C D_x^{\gamma} Y(x_0), {}_a^C D_x^{2\gamma} Y(x_0))^T \text{ and } \mathbf{\Phi}_i := \left(\frac{\Lambda_1^i}{\Gamma(\gamma+1)}, \frac{\Lambda_2^i}{\Gamma(2\gamma+1)} \right)^T.$$

We now define the weighted residual function

$$B(\mathbf{\Pi_Y}) = \sum_{i=1}^{p}(Y_0 - Y_i + \mathbf{\Phi}_i^T\mathbf{\Pi_Y})^2\varpi_i^2,$$

where $\varpi_i = \varpi(x_0, x_i)$ are non-increasing weighting functions as in [16] and [17]. The condition

$$\frac{\partial B}{\partial\{\mathbf{\Pi_Y}\}} = \mathbf{0}$$

is used to minimize the function B, which leads to

$$\sum_{i=1}^{p}\varpi_i^2\mathbf{\Phi}_i\mathbf{\Phi}_i^T\mathbf{\Pi_Y} = \sum_{i=1}^{p}\varpi_i^2(Y_i - Y_0)\mathbf{\Phi}_i.$$

This relation can be written in the form of a system $Q\mathbf{\Pi_Y} = \mathbf{l}$ where

$$Q := \sum_{i=1}^{p}\varpi_i^2\mathbf{\Phi}_i\mathbf{\Phi}_i^T, \quad \mathbf{l} := \sum_{i=1}^{p}\varpi_i^2(Y_i - Y_0)\mathbf{\Phi}_i. \tag{12}$$

Then, $\mathbf{\Pi_Y} = Q^{-1}\mathbf{l}$.

Remark 1 In [25] we proved that matrix Q is positive definite, so we use the Cholesky decomposition. Although we maintain the notation with Q^{-1} for simplicity.

Notice that the vector with the approximations of the fractional derivatives can be explicitly given as

$$\begin{aligned} {}_a^C D_x^{\gamma} Y_0 &= -\nu_{0,1}Y_0 + \sum_{i=1}^{p}\nu_{i,1}Y_i + \mathcal{O}(\Lambda_2^i), \\ {}_a^C D_x^{2\gamma} Y_0 &= -\nu_{0,2}Y_0 + \sum_{i=1}^{p}\nu_{i,2}Y_i + \mathcal{O}(\Lambda_2^i), \end{aligned} \tag{13}$$

that is to say, we can put the value of the fractional derivatives in terms of the value of the solution at the surrounding nodes in a simple way.

Similarly for the right Caputo derivative:

$$\begin{aligned} {}_x^C D_b^{\gamma} Y_0 &= -\eta_{0,1}Y_0 + \sum_{i=1}^{p}\eta_{i,1}Y_i + \mathcal{O}(\Lambda_2^i), \\ {}_x^C D_b^{2\gamma} Y_0 &= -\eta_{0,2}Y_0 + \sum_{i=1}^{p}\eta_{i,2}Y_i + \mathcal{O}(\Lambda_2^i), \end{aligned} \tag{14}$$

By employing a technique quite similar, we can find the GFD approximation of the spatial integer-order derivatives (as discussed in [12] and [23]).

$$\begin{aligned}\frac{\partial Y_0}{\partial x} &= -\sigma_{0,1}Y_0 + \sum_{i=1}^{p}\sigma_{i,1}Y_i + \mathcal{O}(h_i^2),\\ \frac{\partial^2 Y_0}{\partial x^2} &= -\sigma_{0,2}Y_0 + \sum_{i=1}^{p}\sigma_{i,2}Y_i + \mathcal{O}(h_i^2),\end{aligned} \tag{15}$$

for some explicit coefficients and where $h_i = x_i - x_0$.

Remark 2 As shown in [25], the order of the approximation ceasily adjusted by incorporating additional terms into the Taylor polynomial.

3 Numerical scheme

For simplicity, we set $\delta_{j,i} = \nu_{j,i} + \eta_{j,i}$. With this, we arrive at the explicit finite difference scheme

$$\begin{aligned}\epsilon\frac{Y_0^{n+1} - Y_0^n}{\Delta t} &= \frac{1}{2\cos(\frac{\gamma\pi}{2})}\left(-\delta_{0,1}Y_0 + \sum_{i=1}^{p}\delta_{i,1}Y_i\right) + C_0^n + G_0^n\\ &+ D_{1,0}^n\left(-\sigma_{0,1}Y_0 + \sum_{i=1}^{p}\sigma_{i,1}Y_i\right)\\ &+ D_{2,0}^n\left(-\sigma_{0,2}Y_0 + \sum_{i=1}^{p}\sigma_{i,2}Y_i\right) + \mathcal{O}(\Delta t, h_i^2, \Lambda_2^i)\end{aligned} \tag{16}$$

where $D_{1,0}^n = d_1(Y_0^n, n\Delta t, x_0)$, $D_{2,0}^n = d_2(Y_0^n, n\Delta t, x_0)$, $C_0^n = c(Y_0^n, n\Delta t, x_0)$ and

$$G_0^n = \frac{1}{2\cos(\frac{\gamma\pi}{2})}\sum_{k=0}^{n-1}\frac{Y_a^{k,n}(x_0-a)^{k-\gamma} + Y_b^{k,n}(b-x_0)^{k-\gamma}}{\Gamma(k-\gamma+1)}.$$

Notice that the convergence order of (16) is of first order in time and $\min\{2, 2\gamma\}$ in space.

Theorem 1 *The approximations of* ${}_a^C D_x^{\gamma}$, ${}_a^C D_x^{2\gamma}$, ${}_x^C D_b^{2\gamma}$, ${}_x^C D_b^{2\gamma}$, $\frac{\partial}{\partial x}$ *and* $\frac{\partial^2}{\partial x^2}$ *given by (13), (14) and (15) are consistent.*

The proof of the consistency for the integer order can be found in [12] and [11]. For the non-fractional cases, the proof is similar to that in [25]. For the following result, we restrict ourselves to the case $1 \le \gamma < 2$ to avoid complicating the notation in the subsequent discussion, although generalization is straightforward. It's important to note that this implies that

$$G_0^n = \frac{1}{2\cos(\frac{\gamma\pi}{2})}\Bigg[\frac{Y_a^n(x_0-a)^{-\gamma}+Y_b^n(b-x_0)^{-\gamma}}{\Gamma(-\gamma+1)} + \frac{(-\sigma_{a,1}Y_a^n+\sum_{i=1}^p \sigma_{i,1}Y_i^n)(x_0-a)^{1-\gamma}}{\Gamma(2-\gamma)} + \frac{(-\sigma_{b,1}Y_b^n+\sum_{i=1}^p \sigma_{i,1}Y_i^n)(b-x_0)^{1-\gamma}}{\Gamma(2-\gamma)}\Bigg] \tag{17}$$

Theorem 2 *Assuming $\epsilon = 1$ and d_1, d_2 and c are differentiable functions with respect to y, the finite difference scheme (16) is convergent under the condition that*

$$\begin{cases} K_1\delta_{0,1}+K_2\sigma_{0,1}+K_3\sigma_{0,2}-K_4>0 \\ \Delta t < \dfrac{2}{K_1\delta_{0,1}+K_2\sigma_{0,1}+K_3\sigma_{0,2}+K_4}, \end{cases}$$

where K_1, K_2, K_3 and K_4 are defined in the proof.

Proof The analytical solution at $(n\Delta t, x_0)$, y_0^n, also fulfills (16). We substract the expressions for the continuous and discrete solutions and call $e_i^n = |Y_i^n - y_i^n|$. Thanks to the mean value theorem, $C_0^n - c_0^n = \frac{\partial c}{\partial y}e_0^n$; $-D_{1,0}^n Y_0^n + d_{1,0}^n y_0^n \pm D_{1,0}^n y_0^n = -D_{1,0}^n e_0^n - \frac{\partial d_1}{\partial y} e_0^n$ and

$$D_{1,0}^n \sum_{i=1}^p \sigma_{i,1}Y_i^n - d_{1,0}^n \sum_{i=1}^p \sigma_{i,1}y_i^n \pm d_{1,0}^n \sum_{i=1}^p \sigma_{i,1}Y_i^n = \frac{\partial d_1}{\partial y} e_0^n \sum_{i=1}^p \sigma_{i,1}Y_i^n + d_{1,0}^n \sum_{i=1}^p \sigma_{i,1}e_i^n.$$

Then, it is verified

$$\begin{aligned} e_0^{n+1} =& e_0^n + \frac{\Delta t}{2\cos(\frac{\gamma\pi}{2})}\left(-\delta_{0,1}e_0^n + \sum_{i=1}^p \delta_{i,1}e_i^n\right) + \Delta t\frac{\partial c}{\partial y}e_0^n \\ &+ \Delta t \frac{e_a^n(x_0-a)^{-\gamma}+e_b^n(b-x_0)^{-\gamma}}{\Gamma(1-\gamma)} \\ &+ \Delta t\Bigg[-D_{1,0}^n\sigma_{0,1}e_0^n + \frac{\partial d_1}{\partial y}e_0^n\sum_{i=1}^p \sigma_{i,1}Y_i^n + d_{1,0}^n\sum_{i=1}^p \sigma_{i,1}e_i^n \\ &- D_{2,0}^n\sigma_{0,2}e_0^n + \frac{\partial d_2}{\partial y}e_0^n\sum_{i=1}^p \sigma_{i,2}Y_i^n + d_{2,0}^n\sum_{i=1}^p \sigma_{i,2}e_i^n\Bigg] \\ &+ \Delta t\frac{(-\sigma_{a,1}e_a^n+\sum_{i=1}^p \sigma_{i,1}e_i^n)(x_0-a)^{1-\gamma}}{\Gamma(2-\gamma)} \\ &+ \Delta t\frac{(-\sigma_{b,1}e_b^n+\sum_{i=1}^p \sigma_{i,1}e_i^n)(b-x_0)^{1-\gamma}}{\Gamma(2-\gamma)} + \mathcal{O}(\Delta t, h_i^2, \Lambda_2^i) \end{aligned} \tag{18}$$

Now, we take $e^n = \max_{j\in Z}\{e_j^n\}$, so we have that

$$\begin{aligned} e^{n+1} \leq e^n \Bigg[\Bigg| 1 + \Delta t \Bigg(-\frac{\delta_{0,1}}{2\cos(\frac{\gamma\pi}{2})} + \frac{\partial c}{\partial y} - D_{1,0}^n \sigma_{0,1} + \frac{\partial d_1}{\partial y} \sum_{i=1}^{p} \sigma_{i,1} Y_i^n \\ - D_{2,0}^n \sigma_{0,2} + \frac{\partial d_2}{\partial y} \sum_{i=1}^{p} \sigma_{i,2} Y_i^n \Bigg) \Bigg| + \Delta t \Bigg(\frac{1}{2\cos(\frac{\gamma\pi}{2})} \sum_{i=1}^{p} |\delta_{i,1}| \\ + |d_{1,0}^n \sum_{i=1}^{p} \sigma_{i,1}| + |d_{2,0}^n \sum_{i=1}^{p} \sigma_{i,2}| \\ + \frac{(|-\sigma_{a,1}| + |\sum_{i=1}^{p} \sigma_{i,1}|)(x_0 - a)^{1-\gamma}}{\Gamma(2-\gamma)} \\ + \frac{(|-\sigma_{b,1}| + |\sum_{i=1}^{p} \sigma_{i,1}|)(b - x_0)^{1-\gamma}}{\Gamma(2-\gamma)} + \frac{(x_0 - a)^{-\gamma} + (b - x_0)^{-\gamma}}{\Gamma(1-\gamma)} \Bigg) \Bigg]. \end{aligned} \quad (19)$$

We can write the last expression as

$$e^{n+1} \leq e^n [|1 - \Delta t(K_1\delta_{0,1} + K_2\sigma_{0,1} + K_3\sigma_{0,2})| + \Delta t K_4].$$

Hence, the statement of the theorem holds by imposing

$$|1 - \Delta t(K_1\delta_{0,1} + K_2\sigma_{0,1} + K_3\sigma_{0,2})| + \Delta t K_4 < 1.$$

We also provide a result concerning the stability of the scheme.

Theorem 3 *Assume $\epsilon = 1$ and d_1, d_2 and c are y-differentiable functions. Then, the finite difference scheme (16) is conditionally stable.*

Proof We apply the von Neumann analysis and consider the equation for the error, (18). Assume $e_0^n = \xi^n e^{i\vec{k}\vec{r_0}}$, $e_j^n = \xi^n e^{i\vec{k}\vec{r_j}} = \xi^n e^{i\vec{k}(\vec{r_0}+\vec{\theta_j} r)}$, where ξ is an amplification factor, $\vec{k}$ is the wave number and $\vec{r_0}$ and $\vec{r_j}$ are the vectors of coordinates of the central node and the surrounding nodes, respectively. Then, by substitution (we omit the truncation error term),

$$\begin{aligned}\xi^{n+1}e^{i\vec{k}\vec{r_0}} &= \xi^n e^{i\vec{k}\vec{r_0}}\Bigg[1-\frac{\Delta t}{2\cos(\frac{\gamma\pi}{2})}\delta_{0,1}+\Delta t\frac{\partial c}{\partial y}-\Delta t D^n_{1,0}\sigma_{0,1}\\
&\quad+\Delta t\frac{\partial d_1}{\partial y}\sum_{i=1}^{p}\sigma_{i,1}Y_i^n-\Delta t D^n_{2,0}\sigma_{0,2}+\Delta t\frac{\partial d_2}{\partial y}\sum_{i=1}^{p}\sigma_{i,2}Y_i^n\Bigg]\\
&\quad+\frac{\Delta t}{2\cos(\frac{\gamma\pi}{2})}\left(\sum_{i=1}^{p}\delta_{i,1}\xi^n e^{i\vec{k}\vec{r_i}}\right)+\Delta t\xi^n\frac{e^{i\vec{k}\vec{r_a}}(x_0-a)^{-\gamma}+e^{i\vec{k}\vec{r_b}}(b-x_0)^{-\gamma}}{\Gamma(1-\gamma)}\\
&\quad+\Delta t\left[d^n_{1,0}\sum_{i=1}^{p}\sigma_{i,1}\xi^n e^{i\vec{k}\vec{r_i}}+d^n_{2,0}\sum_{i=1}^{p}\sigma_{i,2}\xi^n e^{i\vec{k}\vec{r_i}}\right]\\
&\quad+\Delta t\xi^n\frac{(-\sigma_{a,1}e^{i\vec{k}\vec{r_a}}+\sum_{i=1}^{p}\sigma_{i,1}e^{i\vec{k}\vec{r_i}})(x_0-a)^{1-\gamma}}{\Gamma(2-\gamma)}\\
&\quad+\Delta t\xi^n\frac{(-\sigma_{b,1}e^{i\vec{k}\vec{r_b}}+\sum_{i=1}^{p}\sigma_{i,1}e^{i\vec{k}\vec{r_i}})(b-x_0)^{1-\gamma}}{\Gamma(2-\gamma)}.\end{aligned} \tag{20}$$

We cancel the factor $\xi^n e^{i\vec{k}\vec{r_0}}$, so we have

$$\begin{aligned}\xi &= \Bigg[1-\frac{\Delta t}{2\cos(\frac{\gamma\pi}{2})}\delta_{0,1}+\Delta t\frac{\partial c}{\partial y}-\Delta t D^n_{1,0}\sigma_{0,1}\\
&\quad+\Delta t\frac{\partial d_1}{\partial y}\sum_{i=1}^{p}\sigma_{i,1}Y_i^n-\Delta t D^n_{2,0}\sigma_{0,2}+\Delta t\frac{\partial d_2}{\partial y}\sum_{i=1}^{p}\sigma_{i,2}Y_i^n\Bigg]\\
&\quad+\frac{\Delta t}{2\cos(\frac{\gamma\pi}{2})}\left(\sum_{i=1}^{p}\delta_{i,1}e^{i\vec{k}\vec{\theta_i}r}\right)+\Delta t\frac{e^{i\vec{k}\vec{\theta_a}r}(x_0-a)^{-\gamma}+e^{i\vec{k}\vec{\theta_b}r}(b-x_0)^{-\gamma}}{\Gamma(1-\gamma)}\\
&\quad+\Delta t\left[d^n_{1,0}\sum_{i=1}^{p}\sigma_{i,1}e^{i\vec{k}\vec{\theta_i}r}+d^n_{2,0}\sum_{i=1}^{p}\sigma_{i,2}e^{i\vec{k}\vec{\theta_i}r}\right]\\
&\quad+\Delta t\frac{(-\sigma_{a,1}e^{i\vec{k}\vec{\theta_a}r}+\sum_{i=1}^{p}\sigma_{i,1}e^{i\vec{k}\vec{\theta_i}r})(x_0-a)^{1-\gamma}}{\Gamma(2-\gamma)}\\
&\quad+\Delta t\frac{(-\sigma_{b,1}e^{i\vec{k}\vec{\theta_b}r}+\sum_{i=1}^{p}\sigma_{i,1}e^{i\vec{k}\vec{\theta_i}r})(b-x_0)^{1-\gamma}}{\Gamma(2-\gamma)}.\end{aligned} \tag{21}$$

The stability condition is derived from $|\xi|\leq 1$, and the form of this condition resembles the expression provided in Theorem 2.

4 Numerical examples

This section demonstrates the effectiveness and accuracy of the proposed method through various examples on irregular grids. The method is tested for both dynamic ($\epsilon = 1$) and stationary ($\epsilon = 0$) problems.

4.1 Example 1

Our first example is dedicated to the following fractional differential equation

$$\begin{cases} \dfrac{\partial y}{\partial t} = 0.25(-\Delta)^{\frac{1.8}{2}} y, & x \in [0, \pi], \quad t > 0, \\ y(t, 0) = 0, \quad y(t, 1) = 0, & t > 0, \\ y(0, x) = x^2(\pi - x), & x \in [0, \pi], \end{cases} \tag{22}$$

taken from [28]. Plots of the numerical solution for $t = 0.5$, 1.5, 2.5 and 3.5 s are given in Figure 1 togehter with the maximum error, computed with the analytic solution

$$y(x, t) = \sum_{n=1}^{\infty} \left[\frac{8}{n^3}(-1)^{n+1} - \frac{4}{n^3} \right] \sin(nx) \exp(-0.25(n^2)^{\frac{1.8}{2}} t).$$

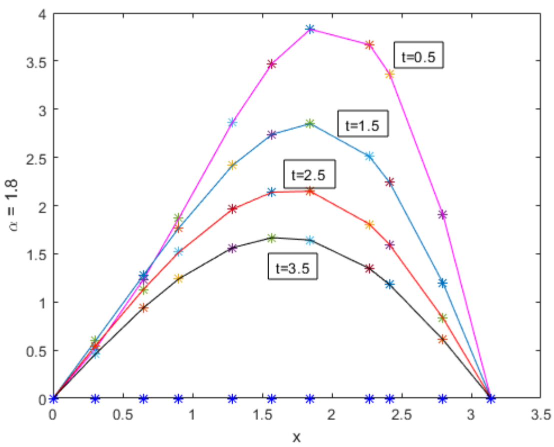

Fig. 1 Solution for times $t = 0.5$, 1.5, 2.5 and 3.5 s. The non-uniform grid can be seen in the x-axis.

$t(s)$	error
0.5	$9.0417 \cdot 10^{-2}$
1.5	$4.2822 \cdot 10^{-2}$
2.5	$8.2390 \cdot 10^{-3}$
3.5	$3.6439 \cdot 10^{-3}$

Table 1 Error for the example 1.

4.2 Example 2

Now, for $\gamma = 0.9$, we perform the finite difference analysis for the example 4.1 of [13], which reads:

$$\begin{cases} \dfrac{\partial y}{\partial t} = -(-\Delta)^{\frac{0.9}{2}} y + (1-t)\pi^{0.9} \sin(\pi x) - \sin(\pi x), & x \in [0,1], \\ y^0(x) = \sin(\pi x), & \\ y(0) = 0, \quad y(1) = 0, & t > 0. \end{cases} \tag{23}$$

The exact solution is given by $y(t,x) = (1-t)\sin(\pi x)$. In Figure 2, the numerical and exact solutions are represented, computed on a non-uniform grid with 15 nodes. Table 2 collects the maximum errors for this example at $t = 0.5, 1.5, 2.5$ and 3.5 s.

$t(s)$	error
0.5	$1.0793 \cdot 10^{-2}$
1.5	$3.8233 \cdot 10^{-3}$
2.5	$9.7921 \cdot 10^{-3}$
3.5	$1.5580 \cdot 10^{-4}$

Table 2 Error for Example 2.

4.3 Example 3:

Now the method is tested in the case $\epsilon = 0$. We look at the Extended Dirichlet problem of [14] for $\gamma = 0.9$:

$$\begin{cases} (-\Delta)^{\frac{0.9}{2}} y = 1, & x \in (-1,1), \\ y = 0, & x \in (-\infty,-1] \cup [1,\infty). \end{cases} \tag{24}$$

The exact solution is given in the cited paper. In Figure 3 we plot the exact and numerical solution using a grid with 23 nodes. The maximum error is $8.8172 \cdot 10^{-2}$.

5 Conclusions

By representing the fractional Laplacian through the Riesz derivative, we were able to derive a simple finite difference scheme that accurately captures the non-local behavior of the operator. In particular, we have shown that the proposed method can accurately approximate the fractional Laplacian for both evolution and stationary problems. Moreover, we have also analyzed the convergence of the method and demonstrated its high degree of accuracy and

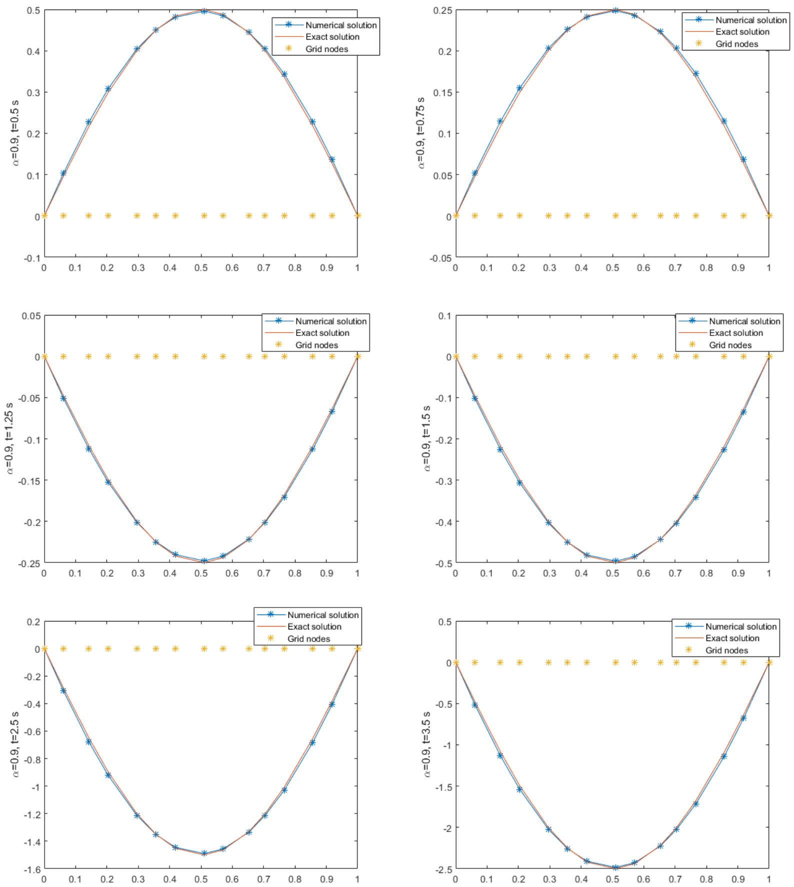

Fig. 2 Solution of Example 2 for time $= 0.5, 0.75, 1.25, 1.5, 2.5$ and 3.5 s. The non-uniform grid can be seen in the x-axis.

stability. The examples considered show that the scheme accurately reproduces continuous behavior for different fractional derivative values, without the constraint of regularly spaced point grids. In conclusion, the finite difference method on non-uniform grids presented in this study represents a reliable and efficient tool for approximating the fractional Laplacian in a wide range of applications. Further research is needed to extend the method to more complex problems and to enhance its computational efficiency and accuracy.

Conflict of Interest

The author has no competing interests to declare that are relevant to the content of this article.

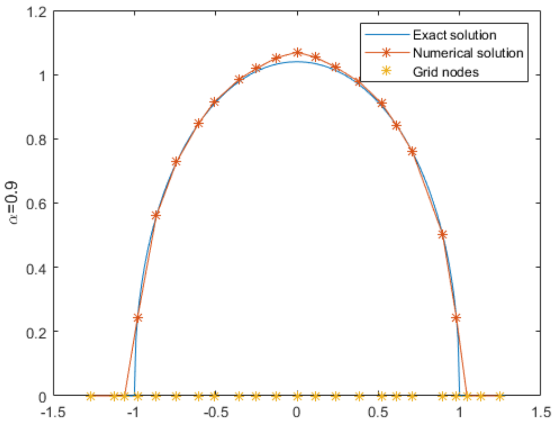

Fig. 3 Solution of Example 3. In blue, the exact solution; the dotted line corresponds to the numerical solution. The non-uniform grid can be seen in the x-axis.

Compliance with Ethical Standards

The paper complies with ethical standards

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